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On a problem by Nathan Jacobson for Malcev algebras

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A B S T R A C T

In this paper, we solve a problem for a certain class of Malcev algebras, which is an analog of an old problem posed by Nathan Jacobson for alternative algebras. Specifically, we prove a coordinatization theorem for a class of Malcev algebras $\mathcal{M}$ that contains the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ such that $m\,\mathfrak{sl}_2(\mathbb{F}) \neq 0$ for any $0 \neq m \in \mathcal{M}$. We drop the last condition and describe the structure of the same class of Malcev algebras $\mathcal{M}$ that contains $\mathfrak{sl}_2(\mathbb{F})$.

1. Introduction

1.1. Overview

A *Malcev algebra* is a $\mathbb{F}$-vector space $\mathcal{M}$ with a bilinear binary operation $(x, y) \mapsto xy$ that satisfies the following identities:

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$$x^2 = 0, \quad J(x, y, xz) = J(x, y, z)x, \tag{1}$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements $x, y, z \in \mathcal{M}$.

Lie algebras fall into the variety of Malcev algebras because the Jacobian of any three elements vanishes. The tangent space $T(L)$ of an analytic Moufang loop L is another example of a Malcev algebra. Let $\mathcal{A}$ be an alternative algebra; if we introduce a new product by means of the commutator $[x, y] = xy - yx$ into $\mathcal{A}$, we obtain a new algebra that will be denoted by $\mathcal{A}^{(-)}$. It is easy to verify that the algebra $\mathcal{A}^{(-)}$ is a Malcev algebra. All Malcev algebras obtained in this form are called *special*. The classic example of a non-Lie Malcev algebra is formed by traceless elements of the Cayley-Dickson algebra with the commutator. This 7-dimensional exceptional Malcev algebra is denoted by $\mathbb{M}$ and is one of most cornerstone examples.

The description of the structure of algebras and superalgebras that contain certain finite-dimensional algebras and superalgebras has a rich history and important applications in representation theory and category theory (for example, see [17,20,23]). The classical Wedderburn Theorem says that if a unital associative algebra $\mathcal{A}$ contains a central simple subalgebra of finite dimension $\mathcal{B}$ with the same identity element, then $\mathcal{A}$ is isomorphic to a Kronecker product $S \otimes_{\mathbb{F}} \mathcal{B}$, where S is the subalgebra of the elements that commute with each $b \in \mathcal{B}$. In particular, if $\mathcal{A}$ contains $M_n(\mathbb{F})$ as a subalgebra with the same identity element, we have $\mathcal{A} \cong M_n(S)$ "coordinated" by S. Kaplansky in Theorem 2 of [18] generalized the Wedderburn result to alternative algebras $\mathcal{A}$ and the split Cayley algebra $\mathcal{B}$. Jacobson in Theorem 1 of [16] gave a new proof of the result of Kaplansky using his classification of completely reducible alternative bimodules over a field of characteristic different of 2 and finally Victor López-Solís in [23] proved that this result is valid for any characteristic. Using this result, Jacobson [16] proved a Kronecker Factorization Theorem for Jordan algebras that contain the Albert algebra with the same identity element. Statements of this type are usually called *Kronecker factorization theorems*. In the case of right alternative algebras, S. Pchelintsev, O. Shashkov, and I. Shestakov [27] proved that every unital right alternative bimodule over a Cayley algebra (over an algebraically closed field of characteristic not 2) is alternative, and they used that result to prove a coordinatization theorem for unital right alternative algebras containing a Cayley subalgebra with the same unit.

In the case of superalgebras, M. López-Díaz and I. Shestakov [20,21] studied the representations of simple alternative and exceptional Jordan superalgebras in characteristic 3 and through these representations, they obtained some analogs of the Kronecker Factorization Theorem for these superalgebras. Also, V. López-Solís in [23] obtained analogs of the Kronecker Factorization Theorem for some simple alternative superalgebras. Similarly, C. Martinez and E. Zelmanov [25] obtained a Kronecker Factorization Theorem for the 10-dimensional exceptional Kac superalgebra K_{10}. Moreover, A. Pozhidaev and I. Shestakov [29] studied the representations of simple finite-dimensional noncommutative Jordan superalgebras and proved some analogs of the Kronecker factorization theorem for such superalgebras (see also [28]).

In regards to Lie theory, [3] S. Berman and R. Moody initiated the study of Lie algebras graded by the root system of characteristic 0 and obtained coordinatization theorems for Lie algebras graded by a simply-laced finite root system of rank $n \geq 2$. Moreover, in [2] G. Benkart and E. Zelmanov determinated some coordinatization theorems for Lie algebras graded by doubly-laced finite root systems (see also Neher [26]).

Carlson in [4] studied Malcev modules for 3-dimensional simple Lie algebras and for the 7-dimensional exceptional Malcev algebra $\mathbb{M}$. She proved that the only non-Lie modules that appear for algebraically closed fields of characteristic $\neq 2, 3$ are a 2-dimensional module for the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ and the regular module for $\mathbb{M}$. In [5,6] it was proved that for fields of characteristic $\neq 2, 3$ any finite-dimensional module over a semisimple algebra which is the direct sum of classical simple Lie algebras and Malcev algebras of type $\mathbb{M}$ splits into a direct sum of a Lie module and a direct sum of irreducible non-Lie modules of the two types mentioned above. Elduque and I. Shestakov in [9] determined the irreducible modules over the Malcev algebras without any restriction on the dimension. The main results obtained by them assert that, essentially, the non-Lie modules that appear in the finite-dimensional case are the only ones in general. So using these results, V. López-Solís in [22] proved an analog of the Kronecker factorization theorem for Malcev (super)algebras that contain (in the even part) $\mathbb{M}$ over fields of characteristic $\neq 2, 3$. Thus, following the study of the structure of (super)algebras that contain (in the even part) finite-dimensional simple algebras, the following question appears: what about the structure of Malcev (super)algebras that contain (in the even part) $\mathfrak{sl}_2(\mathbb{F})$? This question is analogous to the problem posed by Nathan Jacobson for alternative algebras [16].

Analogous to the above problem have been solved for some non-associative algebras. For unital alternative algebras containing a split quaternion algebra with the same identity element has been solved by V. López-Solís and I. Shestakov in [24] (see also [16]). Similarly, for unital Jordan algebras containing the Jordan algebra of symmetric 2×2 matrices has been solved by J. Laliena, V. López-Solís, and I. Shestakov in [19].

1.2. Main result

In this paper, we consider an important subvariety of Malcev algebras. Certainly, Filippov in [10, (6)] introduced a very interesting identity and considered (see [11–14]) the subvariety $\mathcal{H}$ of those Malcev algebras satisfying this identity. He proved in [12] that, in a sense, the subvariety $\mathcal{H}$ plays the same role as the variety of Lie algebras in the structure theory of Malcev algebras. Moreover, the central closures of the prime algebras in $\mathcal{H}$ and the simple algebras in $\mathcal{H}$ are simple algebras of the two types mentioned in the previous subsection (see [7]).

The aim of this paper is to solve the problem proposed in subsection 1.1 for Malcev algebras in the subvariety $\mathcal{H}$, that is, we determine the structure of the Malcev algebras in $\mathcal{H}$ that contain $\mathfrak{sl}_2(\mathbb{F})$.

The paper is organized as follows: In Section 2, we provided some definitions about Malcev algebras, their representations, and graded algebras and modules by the root system. In Section 3, we describe the structure of the Malcev algebras $\mathcal{M}$ in $\mathcal{H}$ that contain $\mathfrak{sl}_2(\mathbb{F})$.

Essentially, the coordinatization Theorem 3.14 involves two ingredients: a Malcev algebra $\mathcal{M}$ in $\mathcal{H}$ containing $\mathfrak{sl}_2(\mathbb{F})$ such that $m\mathfrak{sl}_2(\mathbb{F}) \neq 0$ for any $0 \neq m \in \mathcal{M}$ is "coordinated" by a commutative associative algebra $\mathcal{B}$ and a commutative $\mathcal{B}$-bimodule V, on which a skew-symmetric form is defined with values in $\mathcal{B}$, satisfying Plücker relations. More exactly,

$$\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V^2,$$

with a properly defined multiplication. Moreover, we drop the condition "$m\mathfrak{sl}_2(\mathbb{F}) \neq 0$ for any $0 \neq m \in \mathcal{M}$" and describe the structure of the same class of Malcev algebras $\mathcal{M}$ containing $\mathfrak{sl}_2(\mathbb{F})$.

2. Preliminaries

In this section, we provide background material that will be used along the paper.

Throughout this paper $\mathbb{F}$ will be a field of characteristic different of 2 and 3. So by (1), a Malcev algebra $\mathcal{M}$ is an anticommutative algebra that satisfies the Malcev identity

$$(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z. \tag{2}$$

Here and in what follows, we omit parentheses in left-normed products: $xyzt = ((xy)z)t$, $xy \cdot zt = (xy)(zt)$, $x \cdot yz = x(yz)$, etc. The following functions play an important role in the theory of Malcev algebras:

$$\begin{aligned}
J(x,y,z) &= xyz + yzx + zxy, \textit{ the Jacobian} \text{ of } x,y,z,\\
[x,y,z] &= xyz + x\cdot yz, \ \textit{the antiassociator},\\
\{x,y,z\} &= xyz - xzy + 2x\cdot yz = J(x,y,z) + 3x\cdot yz,\\
h(y,z,t,x,u) &= \{yz,t,u\}x + \{yz,t,x\}u + \{yx,z,u\}t + \{yu,z,x\}t,\\
p(x,y,z,t) &= -\{zt,x,y\} - \{yt,z,x\} + \{xt,y,z\}
\end{aligned}$$

and the operator

$$x\alpha(y,z,t) = p(x,y,z,t).$$

The function p is skew-symmetric in y,z,t. Indeed, the skew symmetry of p in y and z follows from the fact that all Malcev algebras are anticommutative. This implies that α is skew-symmetric on its arguments.

Note that the operator α was defined by V. Filippov in [13] and was essential in [8,9,14,22,30].

2.1. Malcev modules

Recall that $\mathfrak{sl}_2(\mathbb{F})$ is the Lie algebra of traceless 2×2 matrices over $\mathbb{F}$ with the commutator and with the following basis $\{E, H, F\}$ such that

$$EH = E, \;\; FH = -F, \;\; EF = \frac{1}{2}H. \tag{3}$$

This algebra is simple and is one of the main objects of our paper.

Given a Malcev algebra $\mathcal{M}$ and a vector space V over $\mathbb{F}$, V it is said to be a *module* for $\mathcal{M}$ if there exists a $\mathbb{F}$-linear map $\rho : \mathcal{M} \longrightarrow \mathrm{End}_{\mathbb{F}}(V)(x \mapsto \rho_x)$, such that the split extension $\mathscr{E} = \mathcal{M} \oplus V$, with multiplication given by

$$(x + v) \cdot (y + w) := xy + (v\rho_y - w\rho_x) \tag{4}$$

for $x, y \in \mathcal{M}$, $v, w \in V$, is a Malcev algebra. In this case, ρ is called a *representation* of $\mathcal{M}$.

The module V is said to be *irreducible* if $\rho \neq 0$ and it does not contain any proper submodule. V is said to be a Lie module in the case $J(V, \mathcal{M}, \mathcal{M}) = 0$ in $\mathscr{E}$. Also, V is said to be *almost faithful* if $\ker\rho$ does not contain any ideal of $\mathcal{M}$ and faithful if $\ker\rho = 0$.

A $\mathcal{M}$-*adjoint module*, $\overline{\mathcal{M}}$, for algebra $\mathcal{M}$, is defined on the vector space $\mathcal{M}$ with the action of $\mathcal{M}$ coinciding with the multiplication in $\mathcal{M}$. So we have the $\mathfrak{sl}_2(\mathbb{F})$-adjoint module, $\overline{\mathfrak{sl}_2(\mathbb{F})}$.

Example 2.1. The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ has a 2-dimensional non-Lie Malcev module with basis $\{u, v\}$ such that

$$uH = u, \;\; vH = -v, \;\; uE = v, \;\; uF = 0, \;\; vE = 0, \;\; vF = -u.$$

In characteristic 3, this is a Lie module. This $\mathfrak{sl}_2(\mathbb{F})$-module will be called a module of type V_2.

Let V be a module for the Malcev algebra $\mathcal{M}$ and let $\mathscr{E} = \mathcal{M} \oplus V$ be the corresponding split extension. Consider

$$\Gamma = \Gamma(\mathscr{E}) = \{\alpha \in \mathrm{End}_{\mathbb{F}}(\mathscr{E}) : (xy)\alpha = x(y\alpha) = (x\alpha)y \;\; \forall\, x, y \in \mathscr{E}\}, \;\; \textit{the centroid} \text{ of } \mathscr{E},$$
$$Z = Z(\mathscr{E}) = \{\alpha \in \Gamma : \;\; V\alpha \subseteq V \text{ and } \mathcal{M}\alpha \subseteq \mathcal{M}\},$$
$$K = K(V) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) : \; \varphi\rho_x = \rho_x\varphi \;\; \forall\, x \in \mathcal{M}\}, \;\; \textit{the centralizer} \text{ of } V.$$

The following proposition was proved in ([9], Proposition 4) to the superalgebra case. Essentially, it provides some basic properties of the subalgebras Z and K when we consider irreducible almost faithful modules.

Proposition 2.2. *Assume that V is an irreducible almost faithful module for $\mathcal{M}$. Then,*

(i) *Z is an integral domain which acts without torsion on $\mathcal{M}$.*
(ii) *K is a skew field and any nonzero element in K acts bijectively on V.*
(iii) *The restriction homomorphism $\phi : Z \longrightarrow K(\alpha \longmapsto \alpha|_V)$ is one-to-one.*

2.2. Graded algebras and modules by the root system

Let G be any group with group unit 1. If A is an algebra, we say that A is a *G-graded algebra* if there are subspaces A_g, for each $g \in G$, such that

$$A = \bigoplus_{g \in G} A_g \text{ and for each } g, h \in G, A_g A_h \subseteq A_{gh}.$$

If $0 \neq a \in A_g$, we say that a is *homogeneous of G-degree g* and we write $\deg(a) = g$.

We know that the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ has a triangular decomposition $\mathfrak{sl}_2(\mathbb{F}) = L_{-1} \oplus L_0 \oplus L_1$ determined by the Cartan subalgebra $L_0 = \langle H \rangle$, where $L_1 = \langle E \rangle$ and $L_{-1} = \langle F \rangle$. This decomposition is the so-called *the Cartan grading* of $\mathfrak{sl}_2(\mathbb{F})$, from which we conclude that

$$\mathfrak{sl}_2(\mathbb{F}) = \bigoplus_{i \in \mathbf{Z}_3} L_i$$

since $-1 = 2$ in $\mathbf{Z}_3$. Thus $\mathfrak{sl}_2(\mathbb{F})$ is a $\mathbf{Z}_3$-graded Lie algebra.

Definition 2.3. A Lie algebra L over $\mathbb{F}$ is *graded by the root system* Φ, or Φ-graded, if:

1. L contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$ whose root system is Φ, relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $L = \bigoplus_{\alpha \in \Phi \cup \{0\}} L_\alpha$, where $L_\alpha = \{X \in L \mid [X, H] = \alpha(H)X \text{ for all } H \in h\}$; and
3. $L_0 = \sum_{\alpha \in \Phi} [L_\alpha, L_{-\alpha}]$.

The subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of L.

It follows from the Definition 2.3 that $\mathfrak{sl}_2(\mathbb{F})$ is graded by the root system Φ, where $\Phi = \{-1, 1\}$.

Definition 2.4. A Lie module V over a G-graded Lie algebra L is called G-graded if V is a G-graded vector space and

$$L_g V_h \subseteq V_{gh}$$

for all $g, h \in G$.

It is well known that the $\mathfrak{sl}_2(\mathbb{F})$-adjoint module is an irreducible Lie module. As above, we denote this module by $\overline{\mathfrak{sl}_2(\mathbb{F})}$ and, for $a \in \mathfrak{sl}_2(\mathbb{F})$, the corresponding element in $\overline{\mathfrak{sl}_2(\mathbb{F})}$ will be denoted by $\overline{a}$. Therefore, the action of $\mathfrak{sl}_2(\mathbb{F})$ in $\overline{\mathfrak{sl}_2(\mathbb{F})}$ satisfies

$$\begin{array}{lll} E\overline{E} = 0, & E\overline{H} = \overline{E}, & E\overline{F} = \dfrac{1}{2}\overline{H}, \\ H\overline{E} = -\overline{E}, & H\overline{H} = 0, & H\overline{F} = \overline{F}, \\ F\overline{E} = -\dfrac{1}{2}\overline{H}, & F\overline{H} = -\overline{F}, & F\overline{F} = 0. \end{array}$$

It is clear that $\overline{\mathfrak{sl}_2(\mathbb{F})}$ is a $\mathbf{Z}_3$-graded module; $\overline{\mathfrak{sl}_2(\mathbb{F})} = V_{-1} \oplus V_0 \oplus V_1$, where $V_1 = \langle \overline{E} \rangle$, $V_0 = \langle \overline{H} \rangle$ and $V_{-1} = \langle \overline{F} \rangle$.

3. Malcev algebras that contain $\mathfrak{sl}_2(\mathbb{F})$

Let $H(\mathcal{M})$ be the subspace spanned by the elements $h(y, z, t, u, x)$; $H(\mathcal{M})$ is an ideal of the Malcev algebra $\mathcal{M}$ (see [12]). The subvariety $\mathcal{H}$ (over $\mathbb{F}$) is defined as the class of Malcev algebras that satisfy the identity $h(y, z, t, u, x) = 0$, that is, $H(\mathcal{M}) = 0$. In [13] it was proved that a Malcev algebra $\mathcal{M}$ belongs to $\mathcal{H}$ if and only if $\alpha(\mathcal{M}, \mathcal{M}, \mathcal{M})$ (the linear span of the operators $\alpha(x, y, z)$) is contained in $\Gamma(\mathcal{M})$ and in this case $\alpha(\mathcal{M}, \mathcal{M}, \mathcal{M})$ is an ideal of the centroid.

The most important finite-dimensional simple algebras in the variety $\mathcal{H}$ are the 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ and the 7-dimensional exceptional Malcev algebra $\mathbb{M}$ over its centroid Γ, which is a field (see [8], Theorems 2 and 4).

Let $\mathcal{M}$ be a Malcev algebra in $\mathcal{H}$ and let V be a module for $\mathcal{M}$, V is said to be a module for $\mathcal{M}$ in the variety $\mathcal{H}$ or *$\mathcal{H}$-module* if there exists a $\mathbb{F}$-linear map $\rho : \mathcal{M} \longrightarrow \mathrm{End}_{\mathbb{F}}(V)(x \mapsto \rho_x)$ such that the split extension $\mathscr{E} = \mathcal{M} \oplus V$ is again a Malcev algebra in $\mathcal{H}$ with multiplication (4).

Analogously, we can define the subvariety $\mathcal{H}$ to the superalgebras case, which has the same notation as in the algebras. The following results were proved by A. Elduque ([8], Theorem 5 and Corollary 6) to Malcev superalgebras in $\mathcal{H}$ and its proofs can be easily adapted to algebra case.

Theorem 3.1. *Let $\mathcal{M}$ be a simple algebra in the variety $\mathcal{H}$ and let V be a $\mathcal{H}$-module for $\mathcal{M}$. Then V is completely reducible.*

As a consequence of Theorem 3.1, we can deduce the following.

Corollary 3.2. *Let $\mathcal{M}$ be a simple algebra in $\mathcal{H}$, V an $\mathcal{H}$-module for $\mathcal{M}$, and Γ the centroid of $\mathcal{M}$. Then $V = N_V \oplus J_V$ where the submodules N_V and J_V are given by $N_V = \{v \in V : J(v, \mathcal{M}, \mathcal{M}) = 0\}$ and $J_V = J(V, \mathcal{M}, \mathcal{M})$. Moreover,*

(i) *If* $\dim_\Gamma \mathcal{M} = 3$*, then* $N_V = \mathrm{Ann}_V\mathcal{M} \oplus \widehat{N}_V$*, with* $\mathrm{Ann}_V\mathcal{M} = \{v \in V : v\mathcal{M} = 0\}$*,* $\widehat{N}_V = V\mathcal{M} \cap N_V$*. Besides,* $\widehat{N}_V$ *is a direct sum of copies of the adjoint module for* $\mathcal{M}$ *and* J_V *is a direct sum of copies of the unique irreducible non-Lie module for* $\mathcal{M}$*.*

(ii) *If* $\dim_\Gamma \mathcal{M} = 7$*, then* $N_V\mathcal{M} = 0$ *and* J_V *is a direct sum of copies of the adjoint module for* $\mathcal{M}$*.*

Moreover, if $\mathcal{M} = \mathfrak{sl}_2(\mathbb{F})$ in Corollary 3.2 (i), then I. Shestakov ([30], Lemma 8) proved that $J_V = \{v \in V : \{v, \mathcal{M}, \mathcal{M}\} = 0\}$, that is, J_V satisfies the condition

$$vab - vba + 2v \cdot ab = 0,$$

for any $a, b \in \mathcal{M}$ and $v \in J_V$.

Now, if V is a Malcev module for $\mathfrak{sl}_2(\mathbb{F})$ not necessarily in the variety $\mathcal{H}$, it should be noted that N_V need not be completely reducible (see, for example, [1]). This is the reason why we consider Malcev algebras in $\mathcal{H}$ because by Corollary 3.2 (i) $\widehat{N}_V$ it is completely reducible and is a direct sum of copies of the adjoint module for $\mathfrak{sl}_2(\mathbb{F})$.

But in the finite-dimensional case, the situation is different because it was proved that any finite-dimensional Malcev $\mathfrak{sl}_2(\mathbb{F})$-module splits into a direct sum of a completely reducible Lie module and modules of type V_2. The class of irreducible Lie modules is comprised of modules of type L_m, where m is a non-negative integer and L_m has the basis $\{v_0, v_1, \ldots, v_m\}$ with a multiplication table

$$\begin{gathered}
v_i H = \frac{1}{2}(m - 2i)v_i, \quad i = 0, \ldots, m \\
v_i E = \frac{1}{2}v_{i+1}, \quad i = 0, \ldots, m-1, \quad v_m E = 0 \\
v_i F = \frac{1}{2}(mi - (i-1)i)v_{i-1}, \quad i = 1, \ldots, m, \quad v_0 F = 0.
\end{gathered}$$

In [22] was described the structure of Malcev algebras containing the 7-dimensional exceptional Malcev algebra $\mathbb{M}$. In this paper, we describe the structure of the Malcev algebras $\mathcal{M}$ in the variety $\mathcal{H}$ containing $\mathfrak{sl}_2(\mathbb{F})$.

3.1. The case $mL \neq 0$ for any $0 \neq m \in \mathcal{M}$

Let $\mathcal{M}$ be a Malcev algebra in the variety $\mathcal{H}$ that contains a subalgebra $L = \mathfrak{sl}_2(\mathbb{F})$ such that $mL \neq 0$ for any $0 \neq m \in \mathcal{M}$. Then we can consider $\mathcal{M}$ as a Malcev $\mathcal{H}$-module for L and by Corollary 3.2 (i), $\mathrm{Ann}_{\mathcal{M}}L = 0$, $\widehat{N}_{\mathcal{M}} = N_{\mathcal{M}}$ and

$$\mathcal{M} = N_{\mathcal{M}} \oplus J_{\mathcal{M}},$$

where $N_{\mathcal{M}} = \sum_i \oplus \overline{\mathfrak{sl}_2(\mathbb{F})}_i$ with $\overline{\mathfrak{sl}_2(\mathbb{F})}_i \cong \overline{\mathfrak{sl}_2(\mathbb{F})}$ as $\mathfrak{sl}_2(\mathbb{F})$-modules and $J_{\mathcal{M}} = \sum_i \oplus V_{2i}$, with $V_{2i} \cong V_2$ as $\mathfrak{sl}_2(\mathbb{F})$-modules, where V_{2i} is a 2-dimensional non-Lie Malcev module for $\mathfrak{sl}_2(\mathbb{F})$ which has a basis $\{u_i, v_i\}$ (see Example 2.1) satisfying

$$u_iH = u_i,\ v_iH = -v_i,\ u_iE = v_i,\ u_iF = 0,\ v_iE = 0,\ v_iF = -u_i. \tag{5}$$

The expression $N_{\mathcal{M}} = \sum_i \oplus \overline{\mathfrak{sl}_2(\mathbb{F})}_i$ allows us to conclude that $N_{\mathcal{M}}$ is a $\mathbf{Z}_3$-graded module, that is,

$$N_{\mathcal{M}} = V_{-1} \oplus V_0 \oplus V_1,$$

where $V_1 = \langle \overline{E}_i \rangle$, $V_0 = \langle \overline{H}_i \rangle$ and $V_{-1} = \langle \overline{F}_i \rangle$.

Moreover, by the proof of Theorem 5 of [8], we see that

$$N_{\mathcal{M}} = L\alpha(\mathcal{M}, L, L) = \sum_{0 \neq \alpha \in \alpha(\mathcal{M}, L, L)} \oplus L\alpha$$

is the sum of copies of $\overline{L}$ and $\alpha(\mathcal{M}, L, L)$ is a subspace (the linear span of the operators $\alpha(x, y, z)$, where $x \in \mathcal{M}$ and $y, z \in L$) of the ideal $\alpha(\mathcal{M}, \mathcal{M}, \mathcal{M})$ of $\Gamma(\mathcal{M})$.

In Proposition 3.4 we prove a coordinatization theorem for $N_{\mathcal{M}}$, which means that $N_{\mathcal{M}}$ is isomorphic to $\mathfrak{sl}_2(U)$ as algebras, where U is a certain associative commutative algebra.

3.1.1. $\mathbf{Z}_2$*-grading* $\mathcal{M} = N_{\mathcal{M}} \oplus J_{\mathcal{M}}$

Lemma 3.3. *$N_{\mathcal{M}}$ is a subalgebra of $\mathcal{M}$ and $N_{\mathcal{M}} J_{\mathcal{M}} \subseteq J_{\mathcal{M}}$.*

Proof. We have $N_{\mathcal{M}} = \{m \in \mathcal{M} : J(m, L, L) = 0\}$ and $J_{\mathcal{M}} = \{m \in \mathcal{M} : \{m, L, L\} = 0\}$. Then for any $\alpha, \beta \in \alpha(\mathcal{M}, L, L)$

$$J((L\alpha)(L\beta), L, L) \subseteq J(L^2\alpha\beta, L, L) \subseteq J(L, L, L)\alpha\beta = 0,$$

so $(L\alpha)(L\beta) \subseteq N_{\mathcal{M}}$ and $N_{\mathcal{M}} N_{\mathcal{M}} \subseteq N_{\mathcal{M}}$. Consequently $N_{\mathcal{M}}$ is a subalgebra of $\mathcal{M}$.

Furthermore

$$\{(L\alpha)J_{\mathcal{M}}, L, L\} \subseteq \{(LJ_{\mathcal{M}})\alpha, L, L\} \subseteq \{J_{\mathcal{M}}, L, L\}\alpha = 0,$$

then $(L\alpha)J_{\mathcal{M}} \subseteq J_{\mathcal{M}}$ and $N_{\mathcal{M}} J_{\mathcal{M}} \subseteq J_{\mathcal{M}}$.

Thus the lemma is proved. □

Proposition 3.4. *The subalgebra $N_{\mathcal{M}}$ is isomorphic to $L \otimes_{\mathbb{F}} U$, where U is a unital associative commutative algebra.*

Proof. We have $N_{\mathcal{M}} = L\alpha(\mathcal{M}, L, L)$. Denote $U = \alpha(\mathcal{M}, L, L)$, and we will show that U is the desired subalgebra of $\Gamma(\mathcal{M})$.

We fix arbitrary elements $\alpha, \beta \in U$ and $a, b \in L$. So $a\alpha\beta \in N_{\mathcal{M}}$ because

$$J(a\alpha\beta, L, L) \subseteq J(a\alpha, L, L)\beta = 0,$$

then $a\alpha\beta = \sum_i a_i\alpha_i$ for some $a_i \in L$ and $\alpha_i \in U$. Assume that $w = a\alpha\beta - \sum_i a_i\alpha_i = 0$,

$$w = (\delta E + \lambda F + \gamma H)\alpha\beta - \sum_i (\delta_i E + \lambda_i F + \gamma_i H)\alpha_i = 0, \tag{6}$$

where $a = \delta E + \lambda F + \gamma H$, $a_i = \delta_i E + \lambda_i F + \gamma_i H$ and $\delta, \lambda, \gamma, \delta_i, \lambda_i, \gamma_i \neq 0$ are elements of $\mathbb{F}$. Since $\alpha\beta \in \Gamma(\mathcal{M})$ and by the multiplication table of L, from (6) we get

$$0 = Ew = (\frac{1}{2}\lambda H + \gamma E)\alpha\beta - \sum_i (\frac{1}{2}\lambda_i H + \gamma_i E)\alpha_i,$$

$$0 = E \cdot Ew = (\frac{1}{2}\lambda E)\alpha\beta - \sum_i (\frac{1}{2}\lambda_i E)\alpha_i$$

which implies $E\alpha\beta = E\widetilde{\alpha}$, where $\widetilde{\alpha} = \sum_i \lambda_i\lambda^{-1}\alpha_i \in U$. Then $E(\alpha\beta - \widetilde{\alpha}) = 0$. From this it is easy to see that $F(\alpha\beta - \widetilde{\alpha}) = 0$ and $H(\alpha\beta - \widetilde{\alpha}) = 0$; thus

$$L(\alpha\beta - \widetilde{\alpha}) = 0$$

and $\alpha\beta - \widetilde{\alpha} = 0$. Therefore $\alpha\beta = \widetilde{\alpha} \in U$; $UU \subseteq U$.

We consider $m \in \mathcal{M}$ and $a \in L$, then

$$\begin{aligned}((m\alpha)\beta)a &= (m\alpha)(a\beta) = ((m\alpha)a)\beta = (m(a\alpha))\beta \\ &= (m\beta)(a\alpha) = ((m\beta)a)\alpha = ((m\beta)\alpha)a.\end{aligned}$$

If $[\alpha, \beta] = \alpha\beta - \beta\alpha$, we have

$$(\mathcal{M}[\alpha, \beta])L = 0;$$

so $\mathcal{M}[\alpha, \beta] = 0$ (because $\mathrm{Ann}_{\mathcal{M}} L = 0$) and $[\alpha, \beta]|_{\mathcal{M}} = 0$. In particular, $[\alpha, \beta]|_{W_i} = 0$ for any irreducible component W_i of $\mathcal{M}$, then by a version for algebras of the Proposition 2.2(iii) we get $[\alpha, \beta] = 0$ because $\phi : Z \longrightarrow K$ ($\alpha \longmapsto \alpha|_{W_i}$) is one-to-one. Therefore, $[U, U] = 0$; then U is a unital associative commutative algebra.

Also,

$$\begin{aligned}(a\alpha)(b\beta) &= (a(b\beta))\alpha = ((ab)\beta)\alpha \\ &= (ab)(\beta\alpha) = (ab)(\alpha\beta).\end{aligned}$$

Let $v = E\alpha_1 + F\alpha_2 + H\alpha_3 = 0$, where $\alpha_i \in U$. Then, $0 = Ev = \frac{1}{2}H\alpha_2 + E\alpha_3$ and $0 = E \cdot Ev = \frac{1}{2}E\alpha_2$ which implies $E\alpha_2 = 0$. From this, $F\alpha_2 = 0$ and $H\alpha_2 = 0$; so

$$L\alpha_2 = 0$$

and $\alpha_2 = 0$. Similarly, $\alpha_i = 0$ for $i = 1, 3$. Therefore, $N_{\mathcal{M}} \cong L \otimes_{\mathbb{F}} U$ as algebras.

The proposition is proved. $\square$

Corollary 3.5. *The subalgebra $N_{\mathcal{M}}$ of $\mathcal{M}$ is a Lie algebra.*

Corollary 3.6. *The Lie algebra $N_{\mathcal{M}}$ of $\mathcal{M}$ is $\mathbf{Z}_3$-graded perfect Lie algebra.*

Note that the following result does not require the condition: $mL \neq 0$ for any $0 \neq m \in \mathcal{M}$. Therefore, it has a general figure.

Lemma 3.7. *([15], Lemma 9) Let V_{2i} and V_{2j} be the 2-dimensional non-Lie Malcev modules over L with the standard bases $\{u_i, v_i\}$ and $\{u_j, v_j\}$, respectively; then*

$$\begin{array}{lll} u_i u_j E = -u_i v_j, & u_i v_j E = -\dfrac{1}{2} v_i v_j, & v_i v_j E = 0, \\ u_i u_j F = 0, & u_i v_j F = \dfrac{1}{2} u_i u_j, & v_i v_j F = u_i v_j, \\ u_i u_j H = -u_i u_j, & u_i v_j H = 0, & v_i v_j H = v_i v_j \end{array}$$

and $u_i v_j = v_i u_j$ for all i, j. So, $J^2_{\mathcal{M}}$ is a Lie L-module. In particular, if $i = j$, then $V^2_{2i} = 0$.

Proof. From (5) for all i, j, we have

$$\begin{aligned} -u_i v_j F = u_i v_j \cdot FH &\overset{(2)}{=} u_i F v_j H + F v_j H u_i + v_j H u_i F + H u_i F v_j \\ &= u_j H u_i - v_j u_i F - u_i F v_j = u_j u_i - v_j u_i F; \end{aligned}$$

so

$$u_i v_j F = \frac{1}{2} u_i u_j. \tag{7}$$

Also,

$$\begin{aligned} -v_i v_j F = v_i v_j \cdot FH &\overset{(2)}{=} v_i F v_j H + F v_j H v_i + v_j H v_i F + H v_i F v_j \\ &= -u_i v_j H + u_j H v_i - v_j v_i F + v_i F v_j = -u_i v_j H + u_j v_i + v_i v_j F - u_i v_j \end{aligned}$$

and

$$v_i v_j F = \frac{1}{2}(u_i v_j H - u_j v_i + u_i v_j). \tag{8}$$

Similarly,

$$\begin{aligned}\frac{1}{2}u_i v_j H = u_i v_j \cdot EF &\overset{(2)}{=} u_i E v_j F + E v_j F u_i + v_j F u_i E + F u_i E v_j \\ &= v_i v_j F - u_j u_i E \overset{(8)}{=} \frac{1}{2}(u_i v_j H - u_j v_i + u_i v_j) - u_j u_i E;\end{aligned}$$

hence

$$u_i u_j E = -\frac{1}{2}(u_i v_j + v_i u_j). \tag{9}$$

Moreover,

$$\begin{aligned}u_i u_j E = u_i u_j \cdot EH &\overset{(2)}{=} u_i E u_j H + E u_j H u_i + u_j H u_i E + H u_i E u_j \\ &= v_i u_j H - v_j H u_i + u_j u_i E - u_i E u_j = v_i u_j H + v_j u_i + u_j u_i E - v_i v_j,\end{aligned}$$

then

$$2u_i u_j E = v_i u_j H + v_j u_i - v_i v_j,$$

and using (9)

$$-u_i v_j - v_i u_j = v_i u_j H + v_j u_i - v_i v_j,$$

we get

$$v_i u_j H = 0. \tag{10}$$

Then, by (8), $v_i v_j F = \frac{1}{2}(u_i v_j H - u_j v_i + u_i v_j) = \frac{1}{2}(-u_j v_i + u_i v_j)$. Further,

$$\begin{aligned}v_i v_j F = u_i E v_j F &\overset{(2)}{=} u_i v_j \cdot EF - E v_j F u_i - v_j F u_i E - F u_i E v_j \\ &= \frac{1}{2}u_i v_j H + u_j u_i E \overset{(10)}{=} -u_i u_j E,\end{aligned}$$

so by (9) we have

$$v_i v_j F = -u_i u_j E. \tag{11}$$

Besides,

$$\begin{aligned} u_i v_j E &= u_i H v_j E \overset{(2)}{=} u_i v_j \cdot HE - H v_j E u_i - v_j E u_i H - E u_i H v_j \\ &= -u_i v_j E - v_j E u_i + v_i H v_j \\ &= -u_i v_j E - v_i v_j \end{aligned}$$

and

$$u_i v_j E = -\frac{1}{2} v_i v_j. \tag{12}$$

In the same way

$$\begin{aligned} -u_i v_j &= v_i F \cdot u_j E \overset{(2)}{=} v_i u_j FE + u_j F E v_i + F E v_i u_j + E v_i u_j F \\ &\overset{(7)}{=} -\frac{1}{2} u_j u_i E - \frac{1}{2} H v_i u_j = \frac{1}{2} u_i u_j E - \frac{1}{2} v_i u_j, \end{aligned}$$

then $u_i u_j E = -2u_i v_j + v_i u_j$. So, from (9), $-\frac{1}{2}(u_i v_j + v_i u_j) = -2u_i v_j + v_i u_j$ which implies

$$u_i v_j = v_i u_j. \tag{13}$$

Hence, from (9) and (11), we get

$$v_i v_j F = -u_i u_j E = u_i v_j. \tag{14}$$

Furthermore,

$$\begin{aligned} -u_i u_j F &= v_i F u_j F \overset{(2)}{=} v_i u_j \cdot FF - F u_j F v_i - u_j F v_i F - F v_i F u_j \\ &= -u_i F u_j = 0, \end{aligned}$$

then

$$u_i u_j F = 0. \tag{15}$$

Similarly, we have

$$\begin{aligned} -u_i u_j H &= v_i F u_j H \overset{(2)}{=} v_i u_j \cdot FH - F u_j H v_i - u_j H v_i F - H v_i F u_j \\ &= -v_i u_j F - u_j v_i F - v_i F u_j = u_i u_j \end{aligned}$$

and so

$$u_i u_j H = -u_i u_j. \tag{16}$$

Moreover,

$$v_iv_jH = u_iEv_jH \overset{(2)}{=} u_iv_j \cdot EH - Ev_jHu_i - v_jHu_iE - Hu_iEv_j$$
$$= u_iv_jE + v_ju_iE + u_iEv_j = v_iv_j,$$

implies

$$v_iv_jH = v_iv_j. \tag{17}$$

Further,

$$v_iv_jE = u_iEv_jE \overset{(2)}{=} u_iv_j \cdot EE - Ev_jEu_i - v_jEu_iE - Eu_iEv_j$$
$$= v_iEv_j = 0,$$

then

$$v_iv_jE = 0. \tag{18}$$

If $i = j$, then by (14), $u_iv_i = -u_iu_iE = 0$. Thus $V_{2i}^2 = 0$.

We consider the identity [8]

$$J(tx, y, z) = tJ(x, y, z) + J(t, y, z)x - 2J(t, x, yz) \tag{19}$$

which is valid in every Malcev algebra. Then, for any $u, v \in J_{\mathcal{M}}$ and $a, b \in L$, we have $\{u, a, b\} = \{v, a, b\} = 0$ and from (19)

$$\begin{aligned} J(uv, a, b) &= uJ(v, a, b) + J(u, a, b)v - 2J(u, v, ab) \\ &= u(\{v, a, b\} - 3v \cdot ab) + (\{u, a, b\} - 3u \cdot ab)v \\ &\quad - 2(uv \cdot ab + v(ab) \cdot u + (ab)u \cdot v) \\ &= v(ab) \cdot u + (ab)u \cdot v - 2uv \cdot ab \\ &= \{ab, u, v\}. \end{aligned}$$

Thus,

$$J(uv, a, b) = \{ab, u, v\}. \tag{20}$$

Then, for any i, j we obtain

$$\begin{aligned} \{E, u_i, u_j\} &= Eu_iu_j - Eu_ju_i + 2E \cdot u_iu_j \\ &\overset{(14)}{=} -v_iu_j + v_ju_i + 2u_iv_j \overset{(13)}{=} -u_iv_j - u_iv_j + 2u_iv_j = 0, \\ \{F, u_i, u_j\} &= Fu_iu_j - Fu_ju_i + 2F \cdot u_iu_j \overset{(15)}{=} 0, \\ \{H, u_i, u_j\} &= Hu_iu_j - Hu_ju_i + 2H \cdot u_iu_j \overset{(16)}{=} -u_iu_j + u_ju_i + 2u_iu_j = 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned}
\{E, v_i, v_j\} &= Ev_iv_j - Ev_jv_i + 2E \cdot v_iv_j \overset{(18)}{=} 0, \\
\{F, v_i, v_j\} &= Fv_iv_j - Fv_jv_i + 2F \cdot v_iv_j \\
&\overset{(14)}{=} u_iv_j - u_jv_i - 2u_iv_j \overset{(13)}{=} u_iv_j - v_ju_i - 2u_iv_j = 0, \\
\{H, v_i, v_j\} &= Hv_iv_j - Hv_jv_i + 2H \cdot v_iv_j \overset{(17)}{=} v_iv_j - v_jv_i - 2v_iv_j = 0.
\end{aligned} \tag{22}$$

Also, note that the function $\{x, y, z\}$ is skew-symmetry in the last two variables, then it is only necessary to have the following equalities:

$$\begin{aligned}
\{E, u_i, v_j\} &= Eu_iv_j - Ev_ju_i + 2E \cdot u_iv_j \overset{(12)}{=} -v_iv_j + 2(\frac{1}{2}v_iv_j) = 0, \\
\{F, u_i, v_j\} &= Fu_iv_j - Fv_ju_i + 2F \cdot u_iv_j \overset{(7)}{=} -u_ju_i + 2(-\frac{1}{2}u_iu_j) = 0, \\
\{H, u_i, v_j\} &= Hu_iv_j - Hu_jv_i + 2H \cdot u_iv_j \overset{(10)}{=} -u_iv_j - v_ju_i = 0,
\end{aligned} \tag{23}$$

and similarly

$$\{E, v_i, u_j\} = \{F, v_i, u_j\} = \{H, v_i, u_j\} = 0. \tag{24}$$

Therefore, by the relations (20)-(24), $J^2_{\mathcal{M}}$ is a Lie L-module.

The lemma is proved. □

It follows immediately from Lemma 3.3, Proposition 3.4 and Lemma 3.7 the following result.

Corollary 3.8. *$\mathcal{M} = N_{\mathcal{M}} \oplus J_{\mathcal{M}}$ is a $\mathbf{Z}_2$-graded algebra, where $N_{\mathcal{M}}$ is the even part and $J_{\mathcal{M}}$ is the odd part of the $\mathbf{Z}_2$-grading of $\mathcal{M}$.*

3.1.2. Multiplication in $J_{\mathcal{M}}$

We know that $J_{\mathcal{M}}$ is completely reducible and is the direct sum of modules isomorphic to the 2-dimensional non-Lie Malcev module, that is, $J_{\mathcal{M}} = \sum_i \oplus V_{2i}$, where V_{2i} is a non-Lie 2-dimensional Malcev module for the Lie algebra L. Denote by $V(1)$ and $V(2)$ the subspaces of $J_{\mathcal{M}}$ spanned by elements of type u and v, respectively; then the mappings

$$\begin{aligned}
R_E &: V(1) \longrightarrow V(2), \quad w \mapsto wE, \\
R_{-F} &: V(2) \longrightarrow V(1), \quad \widetilde{w} \mapsto -\widetilde{w}F
\end{aligned}$$

are mutually inverse and establish isomorphisms between $V(1)$ and $V(2)$. Clearly, $J_{\mathcal{M}} = V(1) \oplus V(2)$. Consider $V = V(1)$, then for any $w \in V$, denote $w(1) = w$ and $w(2) =$

$R_E(w)$. Thus $V_2(w) = \mathbb{F} \cdot w(1) + \mathbb{F} \cdot w(2) \cong V_2$. So the relations of (5) can be written as follows:

$$\begin{array}{rclrclrcl} w(1)E & = & w(2), & w(1)H & = & w(1), & w(1)F & = & 0, \\ w(2)E & = & 0, & w(2)H & = & -w(2), & w(2)F & = & -w(1), \end{array} \tag{25}$$

for any $w \in V$.

Proposition 3.9. *For any $u, v \in V$, we have*

$$V_2(u) \cdot V_2(v) = L\langle u, v\rangle$$

where $\langle \cdot, \cdot \rangle : V \times V \longrightarrow U$ is a skew-symmetric bilinear mapping. In particular, $V_2(v)^2 = 0$ for any $v \in V$.

Proof. By Corollary 3.8, $J_{\mathcal{M}}^2 \subseteq N_{\mathcal{M}}$, then there exists $\alpha_E, \alpha_F, \alpha_H \in U$ such that $u(1)v(1) = E\alpha_E + F\alpha_F + H\alpha_H$. So by Lemma 3.7, $0 \overset{(15)}{=} u(1)v(1)F = \frac{1}{2}H\alpha_E + F\alpha_H$ which implies $\alpha_E = \alpha_H = 0$, then $u(1)v(1) = F\alpha_F$. Denote $\alpha = \alpha_F \in U$, then

$$u(1)v(1) = F\alpha.$$

Furthermore, we have

$$\begin{aligned} u(1)v(2) &\overset{(14)}{=} -u(1)v(1)E = -F\alpha E = \frac{1}{2}H\alpha, \\ u(2)v(1) &\overset{(14)}{=} v(1)u(1)E = -F\alpha E = \frac{1}{2}H\alpha, \\ u(2)v(2) &\overset{(12)}{=} -2u(1)v(2)E = -2(\frac{1}{2}H\alpha)E = E\alpha \end{aligned}$$

which proves that $V_2(u) \cdot V_2(v) = L\alpha$.

Finally, denote $\alpha = \langle u, v\rangle$, then

$$V_2(u) \cdot V_2(v) = L\langle u, v\rangle, \quad V_2(v) \cdot V_2(u) = L\langle v, u\rangle.$$

But from (13)

$$u(1)v(2) = -v(1)u(2),$$

then $\frac{1}{2}H\langle u, v\rangle = -\frac{1}{2}H\langle v, u\rangle$, which implies $L(\langle u, v\rangle + \langle v, u\rangle) = 0$. So

$$\langle u, v\rangle = -\langle v, u\rangle.$$

Also $F\langle u, v + w\rangle = u(1)(v + w)(1) = u(1)v(1) + u(1)w(1) = F\langle u, v\rangle + F\langle u, w\rangle$, then $L(\langle u, v + w\rangle - \langle u, v\rangle - \langle u, w\rangle) = 0$ and

$$\langle u, v + w\rangle - \langle u, v\rangle - \langle u, w\rangle = 0.$$

Similarly $\langle u + v, w\rangle - \langle u, v\rangle - \langle u, w\rangle = 0$, which proves that $\langle u, v\rangle$ is a bilinear function of u and v, and the proof is complete. □

Lemma 3.10. *For any $u, v, w, t \in V$ the following identities hold:*

$$w\langle u, v\rangle + u\langle v, w\rangle + v\langle w, u\rangle = 0, \tag{26}$$

$$\langle w, t\rangle\langle u, v\rangle + \langle u, t\rangle\langle v, w\rangle + \langle v, t\rangle\langle w, u\rangle = 0. \tag{27}$$

Proof. Recall that in the proof of Proposition 3.9, we obtained the equalities

$$\begin{aligned} u(1)v(1) &= F\langle u, v\rangle, \\ u(1)v(2) = u(2)v(1) &= \frac{1}{2}H\langle u, v\rangle, \\ u(2)v(2) &= E\langle u, v\rangle, \end{aligned} \tag{28}$$

with $\langle u, v\rangle \in U$ and U is a subalgebra of $\Gamma(\mathcal{M})$. Then $u(1)v(1) \cdot w(2) = F\langle u, v\rangle w(2) = w(1)\langle u, v\rangle$ and

$$\begin{aligned} u(1)v(1) \cdot w(2) &= u(1)v(1) \cdot w(1)E = \\ \overset{(2)}{=} u(1)w(1)v(1)E &+ w(1)v(1)Eu(1) + v(1)Eu(1)w(1) + Eu(1)w(1)v(1) = \\ \overset{(25),(28)}{=} F\langle u, w\rangle v(1)E &+ F\langle w, v\rangle Eu(1) + v(2)u(1)w(1) - u(2)w(1)v(1) = \\ &= -\tfrac{1}{2}H\langle w, v\rangle u(1) + \tfrac{1}{2}H\langle v, u\rangle w(1) - \tfrac{1}{2}H\langle u, w\rangle v(1) = \\ &= \tfrac{1}{2}u(1)\langle w, v\rangle - \tfrac{1}{2}w(1)\langle v, u\rangle + \tfrac{1}{2}v(1)\langle u, w\rangle, \end{aligned}$$

thus $w(1)\langle u, v\rangle = \frac{1}{2}u(1)\langle w, v\rangle - \frac{1}{2}w(1)\langle v, u\rangle + \frac{1}{2}v(1)\langle u, w\rangle$ and

$$w(1)\langle u, v\rangle = u(1)\langle w, v\rangle + v(1)\langle u, w\rangle, \tag{29}$$

which proves (26). Multiplying (26) by the element $t \in V$, we get (27). The lemma is proved. □

Similarly, $-u(2)v(2) \cdot w(1) = -E\langle u, v\rangle w(1) = w(2)\langle u, v\rangle$ and

$$\begin{aligned} -u(2)v(2) \cdot w(1) &= u(2)v(2) \cdot w(2)F = \\ \overset{(2)}{=} u(2)w(2)v(2)F &+ w(2)v(2)Fu(2) + v(2)Fu(2)w(2) + Fu(2)w(2)v(2) = \\ \overset{(25),(28)}{=} E\langle u, w\rangle v(2)F &+ E\langle w, v\rangle Fu(2) - v(1)u(2)w(2) + u(1)w(2)v(2) = \\ &= \tfrac{1}{2}H\langle w, v\rangle u(2) - \tfrac{1}{2}H\langle v, u\rangle w(2) + \tfrac{1}{2}H\langle u, w\rangle v(2) = \\ &= \tfrac{1}{2}u(2)\langle w, v\rangle - \tfrac{1}{2}w(2)\langle v, u\rangle + \tfrac{1}{2}v(2)\langle u, w\rangle, \end{aligned}$$

then $w(2)\langle u, v\rangle = \frac{1}{2}u(2)\langle w, v\rangle - \frac{1}{2}w(2)\langle v, u\rangle + \frac{1}{2}v(2)\langle u, w\rangle$ and

$$w(2)\langle u, v\rangle = u(2)\langle w, v\rangle + v(2)\langle u, w\rangle. \tag{30}$$

Therefore, from (29) and (30), we obtain

$$V_2(w)\langle u, v\rangle \subseteq V_2(u)\langle w, v\rangle + V_2(v)\langle u, w\rangle.$$

The following results were proved in [24].

Corollary 3.11. *Let $\{v_i \mid i \in I\}$ be a basis of the space V and let $u_{ij} = \langle v_i, v_j\rangle \in U$. Then the elements u_{ij} satisfy the Plücker relations*

$$u_{ij} = -u_{ji}, \quad u_{ij}u_{kl} + u_{ik}u_{lj} + u_{il}u_{jk} = 0 \tag{31}$$

Lemma 3.12. *Consider the algebra of polynomials $\mathbb{F}[x_1, \dots, x_n; y_1, \dots, y_n]$, and let $\alpha_{ij} = \det \begin{bmatrix} x_i & y_i \\ x_j & y_j \end{bmatrix} \in \mathbb{F}[x_1, \dots, x_n; y_1, \dots, y_n]$. Then the elements $\alpha_{ij} = -\alpha_{ji}$ satisfy the relations (31). Moreover, the algebra $\mathbb{F}[\alpha_{ij} \mid 1 \le i < j \le n]$ is a free algebra modulo relations (31).*

In the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, consider a basis formed by matrices

$$E = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix},$$

that clearly satisfy the multiplication table of $\mathfrak{sl}_2(\mathbb{F})$ (see Subsection 2.1). Also, recall that by Proposition 3.4, $N_{\mathcal{M}} \cong \mathfrak{sl}_2(U)$, hence $\mathcal{M} = \mathfrak{sl}_2(U) \oplus V(1) \oplus V(2)$.

Proposition 3.13. *Let $X, Y \in \mathcal{M}$, $X = X_L + x(1) + y(2)$, $Y = Y_L + z(1) + t(2)$, where $X_L = \frac{1}{2}\begin{pmatrix} -c & -a \\ b & c \end{pmatrix}$, $Y_L = \frac{1}{2}\begin{pmatrix} -f & -d \\ e & f \end{pmatrix}$, $a, b, c, d, e, f \in U$, $x, y, z, t \in V$. Then the product XY is given by*

$$\begin{aligned} XY = X_LY_L + \frac{1}{2}&\begin{pmatrix} -\frac{1}{2}(\langle x, t\rangle + \langle y, z\rangle) & -\langle y, t\rangle \\ \langle x, z\rangle & \frac{1}{2}(\langle x, t\rangle + \langle y, z\rangle) \end{pmatrix} \\ &+ (fx - ey - cz + bt)(1) + (dx - fy - az + ct)(2), \end{aligned}$$

where X_LY_L is the Lie bracket of X_L and Y_L.

Proof. Directly from the relations (25) and (28). $\square$

We can make the formula defining the product in $\mathcal{M}$ more transparent by using the following notation:

$$(u,v) = u(1) + v(2).$$

So, we have $X = X_L + (x,y)$ and $Y = Y_L + (z,t)$, then

$$XY = X_L Y_L + \frac{1}{2}\begin{pmatrix} -\frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) & -\langle y,t\rangle \\ \langle x,z\rangle & \frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) \end{pmatrix} + (x,y)Y_L + X_L(z,t). \quad (32)$$

In the next subsection, we will prove that Proposition 3.13 in fact describes all Malcev extensions of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ in the variety $\mathcal{H}$.

3.1.3. The main theorem

Let $\mathcal{B}$ be a unital associative commutative algebra over a field of characteristic $\neq 2,3$ and let V be a commutative $\mathcal{B}$-bimodule. Assume that there exists a $\mathcal{B}$-bilinear skew-symmetric mapping $\langle\cdot,\cdot\rangle : V \times V \to \mathcal{B}$ such that $\langle V,V\rangle \subseteq \mathcal{B}$ and formula (26) holds for any $u,v,w \in V$.

Let $\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V^2$, where $\mathfrak{sl}_2(\mathcal{B})$ is the vector space of all matrices “coordinated” by $\mathcal{B}$ having trace zero, $V^2 = \{(u,v) \mid u,v \in V\} \cong V \oplus V$ and

$$\mathcal{B} \text{ a subalgebra of } \Gamma(\mathcal{M}).$$

Let $X, Y \in \mathcal{M}$, $X = X_L + (x,y), Y = Y_L + (z,t)$, where $X_L, Y_L \in \mathfrak{sl}_2(\mathcal{B})$ and $(x,y),(z,t) \in V^2$. Define a product in $\mathcal{M}$ by formula (32):

$$XY = X_L Y_L + \frac{1}{2}\begin{pmatrix} -\frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) & -\langle y,t\rangle \\ \langle x,z\rangle & \frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) \end{pmatrix} + (x,y)Y_L + X_L(z,t),$$

where $X_L Y_L$ is the Lie bracket of X_L and Y_L.

Theorem 3.14. *The algebra $\mathcal{M}$ with the product defined above is a Malcev algebra in $\mathcal{H}$ containing $L = \mathfrak{sl}_2$. Conversely, every Malcev algebra in $\mathcal{H}$ that contains a subalgebra $L \cong \mathfrak{sl}_2$ with $mL \neq 0$ for any $0 \neq m \in \mathcal{M}$ has this form.*

Proof. The second part of the theorem follows from Proposition 3.13 with $\mathcal{B} = U$. For the first part, we begin proving that $\mathcal{M}$ is a Malcev algebra.

Let B be a unital associative algebra and let W be a right B-module such that $W[B,B] = 0$, then in this case $w \cdot b = b \cdot w$ for all $w \in W$, $b \in B$. Assume that there exists a B-bilinear skew-symmetric mapping $\langle\cdot,\cdot\rangle : W \times W \to B$ such that $\langle W,W\rangle \subseteq Z(B)$ and for any $u,v,w \in W$

$$\langle u,v\rangle w + \langle v,w\rangle u + \langle w,u\rangle v = 0.$$

Let $A = M_2(B) \oplus W^2$, where $W^2 = \{(u,v) |\, u,v \in W\} \cong W \oplus W$. Let $X, Y \in A$, $X = X_a + (x,y)$, $Y = Y_a + (z,t)$, where $X_a, Y_a \in M_2(B)$ and $(x,y),(z,t) \in W^2$. Define a product in A by formula:

$$XY = X_aY_a + \frac{1}{4}\begin{pmatrix} -\langle x,t\rangle & -\langle y,t\rangle \\ \langle x,z\rangle & \langle y,z\rangle \end{pmatrix} + (z,t)(X_a)^* + (x,y)Y_a,$$

where $A \mapsto A^*$ is the symplectic involution in $M_2(B)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Using the proof of Theorem 5.1 of [24], it is proven that A is a unital alternative algebra. From this we know that $A^- = (A, [\ ,\ ])$ is a Malcev algebra, where $[\ ,\ ]$ is the Lie bracket. The product in $A^{(-)}$ is:

$$\begin{gathered} [X,Y] = XY - YX = \\ = X_aY_a + \tfrac{1}{4}\begin{pmatrix} -\langle x,t\rangle & -\langle y,t\rangle \\ \langle x,z\rangle & \langle y,z\rangle \end{pmatrix} + (z,t)(X_a)^* + (x,y)Y_a - \\ -Y_aX_a - \tfrac{1}{4}\begin{pmatrix} -\langle z,y\rangle & -\langle t,y\rangle \\ \langle z,x\rangle & \langle t,x\rangle \end{pmatrix} - (x,y)(Y_a)^* - (z,t)X_a = \\ = [X_a, Y_a] + \frac{1}{2}\begin{pmatrix} -\frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) & -\langle y,t\rangle \\ \langle x,z\rangle & \frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) \end{pmatrix} + [(x,y), Y_a] + [X_a, (z,t)]. \end{gathered}$$

Take $\mathcal{B} = Z(B)$ and $R = M_2(\mathcal{B}) \oplus W^2$. Thus we see that $\mathcal{M} \cong R^- \leq A^-$, which shows that $\mathcal{M}$ is isomorphic to a subalgebra of A^-. Then $\mathcal{M}$ is a Malcev algebra.

Finally, we will prove that $\mathcal{M}$ belongs to the variety $\mathcal{H}$. For this, it will be enough to verify that $\alpha(\mathcal{M},\mathcal{M},\mathcal{M})$ is contained in $\Gamma(\mathcal{M})$ (see [13]). It is clear that $\alpha(\mathfrak{sl}_2(\mathcal{B}), \mathfrak{sl}_2(\mathcal{B}), \mathfrak{sl}_2(\mathcal{B}))$ is contained in $\Gamma(\mathcal{M})$. Now as α is skew-symmetric, we consider only the following cases:

The cases $\alpha((u,v),E,F),\ \alpha((u,v),E,H),\ \alpha((u,v),F,H)$

We begin with

$$\begin{gathered} E\alpha((u,v),E,F) = p(E,(u,v),E,F) = \\ = -\{EF, E, (u,v)\} - \{(u,v)F, E, E\} + \{EF, (u,v), E\} = \\ = -\tfrac{1}{2}\{H, E, (u,v)\} + \{v(1), E, E\} + \tfrac{1}{2}\{H, (u,v), E\} = \\ = -\tfrac{1}{2}(HE(u,v) - H(u,v)E + 2H \cdot E(u,v)) + \\ +\tfrac{1}{2}(H(u,v)E - HE(u,v) + 2H \cdot (u,v)E) = \\ = -\tfrac{1}{2}(-E(u,v) - (-u(1) + v(2))E - 2Hu(2)) + \tfrac{1}{2}((-u(1) + v(2))E + \\ +E(u,v) + 2Hu(2)) = \\ = E(u,v) + (-u(1) + v(2))E + 2Hu(2) = -u(2) - u(2) + 2u(2) = 0. \end{gathered}$$

Also,

$$F\alpha((u,v),E,F) = p(F,(u,v),E,F) =$$
$$= -\{EF,F,(u,v)\} - \{(u,v)F,E,F\} + \{FF,(u,v),E\}$$
$$= -\tfrac{1}{2}\{H,F,(u,v)\} + \{v(1),E,F\} =$$
$$= -\tfrac{1}{2}(HF(u,v) - H(u,v)F + 2H\cdot F(u,v)) + v(1)EF - v(1)FE + 2v(1)\cdot EF =$$
$$= -\tfrac{1}{2}(F(u,v) - (-u(1)+v(2))F + 2Hv(1)) + v(2)F + v(1)H =$$
$$= -\tfrac{1}{2}(v(1)+v(1)-2v(1)) - v(1) + v(1) = 0.$$

In the same way,

$$H\alpha((u,v),E,F) = p(H,(u,v),E,F) = -\{EF,H,(u,v)\} - \{(u,v)F,E,H\}+$$
$$+\{HF,(u,v),E\} = -\tfrac{1}{2}\{H,H,(u,v)\} + \{v(1),E,H\} + \{F,(u,v),E\} =$$
$$= -\tfrac{1}{2}(HH(u,v) - H(u,v)H + 2H\cdot H(u,v)) + v(1)EH - v(1)HE + 2v(1)\cdot EH+$$
$$+F(u,v)E - FE(u,v) + 2F\cdot(u,v)E = -\tfrac{1}{2}(-(-u(1)+v(2))H + 2H(-u(1)+v(2)))+$$
$$+v(2)H - v(1)E + 2v(1)E + v(1)E + \tfrac{1}{2}H(u,v) + 2Fu(2) = 0.$$

In addition,

$$x(1)\alpha((u,v),E,F) = p(x(1),(u,v),E,F) = -\{EF,x(1),(u,v)\} - \{(u,v)F,E,x(1)\}+$$
$$+\{x(1)F,(u,v),E\} = -\tfrac{1}{2}\{H,x(1),(u,v)\} - \{v(1),E,x(1)\} =$$
$$= -\tfrac{1}{2}(Hx(1)(u,v) - H(u,v)x(1) + 2H\cdot x(1)(u,v))-$$
$$-(v(1)Ex(1) - v(1)x(1)E + 2v(1)\cdot Ex(1)) =$$
$$= -\tfrac{1}{2}(-x(1)(u,v) - (-u(1)+v(2))x(1) + 2H(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle))-$$
$$-(v(2)x(1) - F\langle v,x\rangle E - 2v(1)x(2)) = \tfrac{1}{2}(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle - F\langle u,x\rangle + \tfrac{1}{2}H\langle v,x\rangle-$$
$$-2F\langle x,u\rangle) - \tfrac{1}{2}H\langle v,x\rangle - \tfrac{1}{2}H\langle v,x\rangle + H\langle v,x\rangle = 0$$

and

$$y(2)\alpha((u,v),E,F) = p(y(2),(u,v),E,F) = -\{EF,y(2),(u,v)\} - \{(u,v)F,E,y(2)\}+$$
$$+\{y(2)F,(u,v),E\} = -\tfrac{1}{2}\{H,y(2),(u,v)\} + \{v(1),E,y(2)\} - \{y(1),(u,v),E\} =$$
$$= -\tfrac{1}{2}(Hy(2)(u,v) - H(u,v)y(2) + 2H\cdot y(2)(u,v))+$$
$$+v(1)Ey(2) - v(1)y(2)E + 2v(1)\cdot Ey(2)-$$
$$-y(1)(u,v)E + y(1)E(u,v) - 2y(1)\cdot(u,v)E =$$
$$= -\tfrac{1}{2}(y(2)(u,v) - (-u(1)+v(2))y(2) + 2H(\tfrac{1}{2}H\langle y,u\rangle + E\langle y,v\rangle))+$$

$$+v(2)y(2) - \tfrac{1}{2}H\langle v,y\rangle E - (F\langle y,u\rangle + \tfrac{1}{2}H\langle y,v\rangle)E + y(2)(u,v) - 2y(1)u(2) =$$
$$= -\tfrac{1}{2}(y(2)u(1) + y(2)v(2) + u(1)y(2) - v(2)y(2) - 2E\langle y,v\rangle)+$$
$$+v(2)y(2) + \tfrac{1}{2}E\langle v,y\rangle + \tfrac{1}{2}H\langle y,u\rangle + \tfrac{1}{2}E\langle y,v\rangle + y(2)u(1) + y(2)v(2) - H\langle y,u\rangle =$$
$$= \tfrac{1}{2}E\langle v,y\rangle + \tfrac{1}{2}H\langle y,u\rangle + \tfrac{1}{2}E\langle y,v\rangle + \tfrac{1}{2}H\langle y,u\rangle - H\langle y,u\rangle = 0.$$

Similarly $\alpha((u,v),E,H)|_{\mathcal{M}} = 0$ and $\alpha((u,v),F,H)|_{\mathcal{M}} = 0$, then obviously $\alpha((u,v),E,F)$, $\alpha((u,v),E,H)$ and $\alpha((u,v),F,H)$ belong to $\Gamma(\mathcal{M})$.

The cases $\alpha((u,v),(z,t),E),\ \alpha((u,v),(z,t),F),\ \alpha((u,v),(z,t),H)$

Now we begin with

$$E\alpha((u,v),(z,t),E) = p(E,(u,v),(z,t),E) =$$
$$= -\{(z,t)E, E, (u,v)\} - \{(u,v)E, (z,t), E\} + \{EE, (u,v), (z,t)\} =$$
$$= -\{z(2), E, (u,v)\} - \{u(2), (z,t), E\} =$$
$$= -z(2)E(u,v) + z(2)(u,v)E - 2z(2)\cdot E(u,v)-$$
$$-u(2)(z,t)E + u(2)E(z,t) - 2u(2)\cdot(z,t)E =$$
$$= (\tfrac{1}{2}H\langle z,u\rangle + E\langle z,v\rangle)E + 2z(2)u(2) - (\tfrac{1}{2}H\langle u,z\rangle + E\langle u,t\rangle)E - 2u(2)z(2) =$$
$$= -\tfrac{1}{2}E\langle z,u\rangle + 2E\langle z,u\rangle + \tfrac{1}{2}E\langle u,z\rangle - 2E\langle u,z\rangle = 3E\langle z,u\rangle.$$

Furthermore,

$$F\alpha((u,v),(z,t),E) = p(F,(u,v),(z,t),E) =$$
$$= -\{(z,t)E, F, (u,v)\} - \{(u,v)E, (z,t), F\} + \{FE, (u,v), (z,t)\} =$$
$$= -\{z(2), F, (u,v)\} - \{u(2), (z,t), F\} - \tfrac{1}{2}\{H, (u,v), (z,t)\} =$$
$$= -z(2)F(u,v) + z(2)(u,v)F - 2z(2)\cdot F(u,v)-$$
$$-u(2)(z,t)F + u(2)F(z,t) - 2u(2)\cdot(z,t)F-$$
$$-\tfrac{1}{2}(H(u,v)(z,t) - H(z,t)(u,v) + 2H\cdot(u,v)(z,t)) =$$
$$= z(1)(u,v) + (\tfrac{1}{2}H\langle z,u\rangle + E\langle z,v\rangle)F - 2z(2)v(1)-$$
$$-(\tfrac{1}{2}H\langle u,z\rangle + E\langle u,t\rangle)F - u(1)(z,t) + 2u(2)t(1)-$$
$$-\tfrac{1}{2}((-u(1)+v(2))(z,t) - (-z(1)+t(2))(u,v)+$$
$$+2H(F\langle u,z\rangle + \tfrac{H}{2}\langle u,t\rangle + \tfrac{H}{2}\langle v,z\rangle + E\langle v,t\rangle)) =$$
$$= F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle + \tfrac{1}{2}F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle - H\langle z,v\rangle-$$
$$-\tfrac{1}{2}F\langle u,z\rangle - \tfrac{1}{2}H\langle u,t\rangle - F\langle u,z\rangle - \tfrac{1}{2}H\langle u,t\rangle + H\langle u,t\rangle-$$
$$-\tfrac{1}{2}(-F\langle u,z\rangle - \tfrac{1}{2}H\langle u,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle+$$

$$+F\langle z,u\rangle+\tfrac{1}{2}H\langle z,v\rangle-\tfrac{1}{2}H\langle t,u\rangle-E\langle t,v\rangle+$$
$$+2F\langle u,z\rangle-2E\langle v,t\rangle)=3F\langle z,u\rangle.$$

Also,

$$H\alpha((u,v),(z,t),E)=p(H,(u,v),(z,t),E)=$$
$$=-\{(z,t)E,H,(u,v)\}-\{(u,v)E,(z,t),H\}+\{HE,(u,v),(z,t)\}=$$
$$=-\{z(2),H,(u,v)\}-\{u(2),(z,t),H\}-\{E,(u,v),(z,t)\}=$$
$$=-(z(2)H(u,v)-z(2)(u,v)H+2z(2)\cdot H(u,v))-$$
$$-(u(2)(z,t)H-u(2)H(z,t)+2u(2)\cdot(z,t)H)-$$
$$-(E(u,v)(z,t)-E(z,t)(u,v)+2E\cdot(u,v)(z,t))=$$
$$=z(2)(u,v)+(\tfrac{1}{2}H\langle z,u\rangle+E\langle z,v\rangle)H-2z(2)(-u(1)+v(2))-$$
$$-(\tfrac{1}{2}H\langle u,z\rangle+E\langle u,t\rangle)H-u(2)(z,t)-2u(2)(z(1)-t(2))+$$
$$+u(2)(z,t)-z(2)(u,v)-2E(F\langle u,z\rangle+\tfrac{H}{2}\langle u,t\rangle+\tfrac{H}{2}\langle v,z\rangle+E\langle v,t\rangle)=$$
$$=\tfrac{1}{2}H\langle z,u\rangle+E\langle z,v\rangle+E\langle z,v\rangle+H\langle z,u\rangle-2E\langle z,v\rangle-$$
$$-E\langle u,t\rangle-\tfrac{1}{2}H\langle u,z\rangle-E\langle u,t\rangle-H\langle u,z\rangle+2E\langle u,t\rangle+$$
$$+\tfrac{1}{2}H\langle u,z\rangle+E\langle u,t\rangle-\tfrac{1}{2}H\langle z,u\rangle-E\langle z,v\rangle-$$
$$-H\langle u,z\rangle-E\langle u,t\rangle-E\langle v,z\rangle=3H\langle z,u\rangle.$$

In addition,

$$x(1)\alpha((u,v),(z,t),E)=p(x(1),(u,v),(z,t),E)=$$
$$=-\{(z,t)E,x(1),(u,v)\}-\{(u,v)E,(z,t),x(1)\}+\{x(1)E,(u,v),(z,t)\}=$$
$$=-\{z(2),x(1),(u,v)\}-\{u(2),(z,t),x(1)\}+\{x(2),(u,v),(z,t)\}=$$
$$=-(z(2)x(1)(u,v)-z(2)(u,v)x(1)+2z(2)\cdot x(1)(u,v))-$$
$$-(u(2)(z,t)x(1)-u(2)x(1)(z,t)+2u(2)\cdot(z,t)x(1))+$$
$$+x(2)(u,v)(z,t)-x(2)(z,t)(u,v)+2x(2)\cdot(u,v)(z,t)$$
$$=-\tfrac{1}{2}H\langle z,x\rangle(u,v)+(\tfrac{1}{2}H\langle z,u\rangle+E\langle z,v\rangle)x(1)-2z(2)(F\langle x,u\rangle+\tfrac{1}{2}H\langle x,v\rangle)-$$
$$-(\tfrac{1}{2}H\langle u,z\rangle+E\langle u,t\rangle)x(1)+\tfrac{1}{2}H\langle u,x\rangle(z,t)-2u(2)(F\langle z,x\rangle+\tfrac{1}{2}H\langle t,x\rangle)+$$
$$+(\tfrac{1}{2}H\langle x,u\rangle+E\langle x,v\rangle)(z,t)-(\tfrac{1}{2}H\langle x,z\rangle+E\langle x,t\rangle)(u,v)+$$
$$+2x(2)(F\langle u,z\rangle+\tfrac{H}{2}\langle u,t\rangle+\tfrac{H}{2}\langle v,z\rangle+E\langle v,t\rangle)=$$
$$=\tfrac{1}{2}u(1)\langle z,x\rangle-\tfrac{1}{2}v(2)\langle z,x\rangle-\tfrac{1}{2}x(1)\langle z,u\rangle-x(2)\langle z,v\rangle+2z(1)\langle x,u\rangle+z(2)\langle x,v\rangle+$$
$$+\tfrac{1}{2}x(1)\langle u,z\rangle+x(2)\langle u,t\rangle-\tfrac{1}{2}z(1)\langle u,x\rangle+\tfrac{1}{2}t(2)\langle u,x\rangle+2u(1)\langle z,x\rangle+u(2)\langle t,x\rangle-$$
$$-\tfrac{1}{2}z(1)\langle x,u\rangle+\tfrac{1}{2}t(2)\langle x,u\rangle-z(2)\langle x,v\rangle+$$

$$+\tfrac{1}{2}u(1)\langle x,z\rangle - \tfrac{1}{2}v(2)\langle x,z\rangle + u(2)\langle x,t\rangle -$$
$$-2x(1)\langle u,z\rangle - x(2)\langle u,t\rangle - x(2)\langle v,z\rangle =$$
$$= 2(u(1)\langle z,x\rangle + z(1)\langle x,u\rangle) - x(1)\langle u,z\rangle \overset{(26)}{=} 3x(1)\langle z,u\rangle$$

and

$$y(2)\alpha((u,v),(z,t),E) = p(y(2),(u,v),(z,t),E) =$$
$$= -\{(z,t)E, y(2),(u,v)\} - \{(u,v)E,(z,t),y(2)\} + \{y(2)E,(u,v),(z,t)\} =$$
$$= -\{z(2),y(2),(u,v)\} - \{u(2),(z,t),y(2)\} =$$
$$= -(z(2)y(2)(u,v) - z(2)(u,v)y(2) + 2z(2)\cdot y(2)(u,v)) -$$
$$-(u(2)(z,t)y(2) - u(2)y(2)(z,t) + 2u(2)\cdot (z,t)y(2)) =$$
$$= -E\langle z,y\rangle(u,v) + (\tfrac{1}{2}H\langle z,u\rangle + E\langle z,v\rangle)y(2) - 2z(2)(\tfrac{1}{2}H\langle y,u\rangle + E\langle y,v\rangle) -$$
$$-(\tfrac{1}{2}H\langle u,z\rangle + E\langle u,t\rangle)y(2) + E\langle u,y\rangle(z,t) - 2u(2)(\tfrac{1}{2}H\langle z,y\rangle + E\langle t,y\rangle) =$$
$$= u(2)\langle z,y\rangle + \tfrac{1}{2}y(2)\langle z,u\rangle + z(2)\langle y,u\rangle - \tfrac{1}{2}y(2)\langle u,z\rangle - z(2)\langle u,y\rangle + u(2)\langle z,y\rangle =$$
$$= 2(u(2)\langle z,y\rangle + z(2)\langle y,u\rangle) + y(2)\langle z,u\rangle \overset{(26)}{=} 3y(2)\langle z,u\rangle.$$

Similarly,

$$E\alpha((u,v),(z,t),F) = 3E\langle t,v\rangle,$$
$$F\alpha((u,v),(z,t),F) = 3F\langle t,v\rangle,$$
$$H\alpha((u,v),(z,t),F) = 3H\langle t,v\rangle,$$
$$x(1)\alpha((u,v),(z,t),F) = 3x(1)\langle t,v\rangle,$$
$$y(2)\alpha((u,v),(z,t),F) = 3y(2)\langle t,v\rangle.$$

Moreover,

$$E\alpha((u,v),(z,t),H) = p(E,(u,v),(z,t),H) =$$
$$= -\{(z,t)H, E,(u,v)\} - \{(u,v)H,(z,t),E\} + \{EH,(u,v),(z,t)\} =$$
$$\overset{(21)-(24)}{=} -\{z(1)-t(2),E,(u,v)\} - \{u(1)-v(2),(z,t),E\} =$$
$$= -\{z(1),E,(u,v)\} + \{t(2),E,(u,v)\} - \{u(1),(z,t),E\} + \{v(2),(z,t),E\} =$$
$$= -z(1)E(u,v) + z(1)(u,v)E - 2z(1)\cdot E(u,v) +$$
$$+t(2)E(u,v) - t(2)(u,v)E + 2t(2)\cdot E(u,v) -$$
$$-u(1)(z,t)E + u(1)E(z,t) - 2u(1)\cdot (z,t)E +$$
$$+v(2)(z,t)E - v(2)E(z,t) + 2v(2)\cdot (z,t)E =$$

$$
\begin{gathered}
= -z(2)(u,v) + (F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle)E + 2z(1)u(2) - \\
-(\tfrac{1}{2}H\langle t,u\rangle + E\langle t,v\rangle)E - 2t(2)u(2) - \\
-(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle)E + u(2)(z,t) - 2u(1)z(2) + \\
+(\tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle)E + 2v(2)z(2) = \\
= -\tfrac{1}{2}H\langle z,u\rangle - E\langle z,v\rangle - \tfrac{1}{2}H\langle z,u\rangle - \tfrac{1}{2}E\langle z,v\rangle + H\langle z,u\rangle + +\tfrac{1}{2}E\langle t,u\rangle - 2E\langle t,u\rangle + \\
+\tfrac{1}{2}H\langle u,z\rangle + \tfrac{1}{2}E\langle u,t\rangle + \tfrac{1}{2}H\langle u,z\rangle + E\langle u,t\rangle - H\langle u,z\rangle - \tfrac{1}{2}E\langle v,z\rangle + 2E\langle v,z\rangle = \\
= 3E(\langle u,t\rangle + \langle v,z\rangle).
\end{gathered}
$$

Further,

$$
\begin{gathered}
F\alpha((u,v),(z,t),H) = p(F,(u,v),(z,t),H) = \\
= -\{(z,t)H, F, (u,v)\} - \{(u,v)H, (z,t), F\} + \{FH, (u,v), (z,t)\} = \\
\overset{(21)-(24)}{=} -\{z(1) - t(2), F, (u,v)\} - \{u(1) - v(2), (z,t), F\} = \\
= -\{z(1), F, (u,v)\} + \{t(2), F, (u,v)\} - \{u(1), (z,t), F\} + \{v(2), (z,t), F\} = \\
= -z(1)F(u,v) + z(1)(u,v)F - 2z(1)\cdot F(u,v) + \\
+t(2)F(u,v) - t(2)(u,v)F + 2t(2)\cdot F(u,v) - \\
-u(1)(z,t)F + u(1)F(z,t) - 2u(1)\cdot(z,t)F + \\
+v(2)(z,t)F - v(2)F(z,t) + 2v(2)\cdot(z,t)F = \\
= (F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle)F - 2z(1)v(1) - \\
-t(1)(u,v) - (\tfrac{1}{2}H\langle t,u\rangle + E\langle t,v\rangle)F + 2t(2)v(1) - \\
-(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle)F + 2u(1)t(1) + \\
+(\tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle)F + v(1)(z,t) - 2v(2)t(1) = \\
= \tfrac{1}{2}F\langle z,v\rangle - 2F\langle z,v\rangle - F\langle t,u\rangle - \tfrac{1}{2}H\langle t,v\rangle - \tfrac{1}{2}F\langle t,u\rangle - \tfrac{1}{2}H\langle t,v\rangle + H\langle t,v\rangle - \\
-\tfrac{1}{2}F\langle u,t\rangle + 2F\langle u,t\rangle + \tfrac{1}{2}F\langle v,z\rangle + \tfrac{1}{2}H\langle v,t\rangle + F\langle v,z\rangle + \tfrac{1}{2}H\langle v,t\rangle - H\langle v,t\rangle = \\
= 3F(\langle u,t\rangle + \langle v,z\rangle).
\end{gathered}
$$

In the same way,

$$
\begin{gathered}
H\alpha((u,v),(z,t),H) = p(H,(u,v),(z,t),H) = \\
= -\{(z,t)H, H, (u,v)\} - \{(u,v)H, (z,t), H\} + \{HH, (u,v), (z,t)\} = \\
= -\{z(1) - t(2), H, (u,v)\} - \{u(1) - v(2), (z,t), H\} = \\
= -\{z(1), H, (u,v)\} + \{t(2), H, (u,v)\} - \{u(1), (z,t), H\} + \{v(2), (z,t), H\} = \\
= -z(1)H(u,v) + z(1)(u,v)H - 2z(1)\cdot H(u,v) +
\end{gathered}
$$

$$+t(2)H(u,v) - t(2)(u,v)H + 2t(2) \cdot H(u,v) -$$
$$-u(1)(z,t)H + u(1)H(z,t) - 2u(1) \cdot (z,t)H +$$
$$+v(2)(z,t)H - v(2)H(z,t) + 2v(2) \cdot (z,t)H =$$
$$= -z(1)(u,v) + (F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle)H - 2z(1)(-u(1)+v(2)) -$$
$$-t(2)(u,v) - (\tfrac{1}{2}H\langle t,u\rangle + E\langle t,v\rangle)H + 2t(2)(-u(1)+v(2)) -$$
$$-(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle)H + u(1)(z,t) - 2u(1)(z(1)-t(2)) +$$
$$+(\tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle)H + v(2)(z,t) + 2v(2)(z(1)-t(2)) =$$
$$= -F\langle z,u\rangle - \tfrac{1}{2}H\langle z,v\rangle - F\langle z,u\rangle + 2F\langle z,u\rangle - H\langle z,v\rangle -$$
$$-\tfrac{1}{2}H\langle t,u\rangle - E\langle t,v\rangle - E\langle t,v\rangle - H\langle t,u\rangle + 2E\langle t,v\rangle +$$
$$+F\langle u,z\rangle + F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle - 2F\langle u,z\rangle + H\langle u,t\rangle +$$
$$+E\langle v,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle + H\langle v,z\rangle - 2E\langle v,t\rangle =$$
$$= 3H(\langle u,t\rangle + \langle v,z\rangle).$$

Following the same process,

$$x(1)\alpha((u,v),(z,t),H) = p(x(1),(u,v),(z,t),H) =$$
$$= -\{(z,t)H, x(1), (u,v)\} - \{(u,v)H, (z,t), x(1)\} + \{x(1)H, (u,v), (z,t)\} =$$
$$= -\{z(1)-t(2), x(1), (u,v)\} - \{u(1)-v(2), (z,t), x(1)\} + \{x(1), (u,v), (z,t)\} =$$
$$= -z(1)x(1)(u,v) + z(1)(u,v)x(1) - 2z(1) \cdot x(1)(u,v) +$$
$$+t(2)x(1)(u,v) - t(2)(u,v)x(1) + 2t(2) \cdot x(1)(u,v) -$$
$$-u(1)(z,t)x(1) + u(1)x(1)(z,t) - 2u(1) \cdot (z,t)x(1) +$$
$$+v(2)(z,t)x(1) - v(2)x(1)(z,t) + 2v(2) \cdot (z,t)x(1) +$$
$$+x(1)(u,v)(z,t) - x(1)(z,t)(u,v) + 2x(1) \cdot (u,v)(z,t) =$$
$$= -F\langle z,x\rangle(u,v) + (F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle)x(1) - 2z(1)(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle) +$$
$$+\tfrac{1}{2}H\langle t,x\rangle(u,v) - (\tfrac{1}{2}H\langle t,u\rangle + E\langle t,v\rangle)x(1) + 2t(2)(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle) -$$
$$-(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle)x(1) + F\langle u,x\rangle(z,t) - 2u(1)(F\langle z,x\rangle + \tfrac{1}{2}H\langle t,x\rangle) +$$
$$+(\tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle)x(1) - \tfrac{1}{2}H\langle v,x\rangle(z,t) + 2v(2)(F\langle z,x\rangle + \tfrac{1}{2}H\langle t,x\rangle) +$$
$$+(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle)(z,t) - (F\langle x,z\rangle + \tfrac{1}{2}H\langle x,t\rangle)(u,v) +$$
$$+2x(1)(F\langle u,z\rangle + \tfrac{1}{2}H(\langle u,t\rangle + \langle v,z\rangle) + E\langle v,t\rangle) =$$
$$= -v(1)\langle z,x\rangle - \tfrac{1}{2}x(1)\langle z,v\rangle - z(1)\langle x,v\rangle +$$
$$+\tfrac{1}{2}(-u(1)+v(2))\langle t,x\rangle + \tfrac{1}{2}x(1)\langle t,u\rangle + x(2)\langle t,v\rangle - 2t(1)\langle x,u\rangle - t(2)\langle x,v\rangle +$$
$$+\tfrac{1}{2}x(1)\langle u,t\rangle + t(1)\langle u,x\rangle - u(1)\langle t,x\rangle -$$
$$-\tfrac{1}{2}x(1)\langle v,z\rangle - x(2)\langle v,t\rangle - \tfrac{1}{2}(-z(1)+t(2))\langle v,x\rangle - 2v(1)\langle z,x\rangle - v(2)\langle t,x\rangle +$$

$$+t(1)\langle x,u\rangle + \tfrac{1}{2}(-z(1)+t(2))\langle x,v\rangle - v(1)\langle x,z\rangle - \tfrac{1}{2}(-u(1)+v(2))\langle x,t\rangle +$$
$$+x(1)(\langle u,t\rangle + \langle v,z\rangle) + 2x(2)\langle v,t\rangle =$$
$$= -2(u(1)\langle t,x\rangle + t(1)\langle x,u\rangle) - 2(v(1)\langle z,x\rangle + z(1)\langle x,v\rangle) + x(1)(\langle u,t\rangle + \langle v,z\rangle) =$$
$$\stackrel{(26)}{=} 3x(1)(\langle u,t\rangle + \langle v,z\rangle)$$

and

$$y(2)\alpha((u,v),(z,t),H) = p(y(2),(u,v),(z,t),H) =$$
$$= -\{(z,t)H, y(2),(u,v)\} - \{(u,v)H,(z,t),y(2)\} + \{y(2)H,(u,v),(z,t)\} =$$
$$= -\{z(1)-t(2), y(2),(u,v)\} - \{u(1)-v(2),(z,t),y(2)\} - \{y(2),(u,v),(z,t)\} =$$
$$= -z(1)y(2)(u,v) + z(1)(u,v)y(2) - 2z(1)\cdot y(2)(u,v) +$$
$$+t(2)y(2)(u,v) - t(2)(u,v)y(2) + 2t(2)\cdot y(2)(u,v) -$$
$$-u(1)(z,t)y(2) + u(1)y(2)(z,t) - 2u(1)\cdot (z,t)y(2) +$$
$$+v(2)(z,t)y(2) - v(2)y(2)(z,t) + 2v(2)\cdot (z,t)y(2) -$$
$$-y(2)(u,v)(z,t) + y(2)(z,t)(u,v) - 2y(2)\cdot (u,v)(z,t) =$$
$$= -\tfrac{1}{2}H\langle z,y\rangle(u,v) + (F\langle z,u\rangle + \tfrac{1}{2}H\langle z,v\rangle)y(2) - 2z(1)(\tfrac{1}{2}H\langle y,u\rangle + E\langle y,v\rangle) +$$
$$+E\langle t,y\rangle(u,v) - (\tfrac{1}{2}H\langle t,u\rangle + E\langle t,v\rangle)y(2) + 2t(2)(\tfrac{1}{2}H\langle y,u\rangle + E\langle y,v\rangle) -$$
$$-(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle)y(2) + \tfrac{1}{2}H\langle u,y\rangle(z,t) - 2u(1)(\tfrac{1}{2}H\langle z,y\rangle + E\langle t,y\rangle) +$$
$$+(\tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle)y(2) - E\langle v,y\rangle(z,t) + 2v(2)(\tfrac{1}{2}H\langle z,y\rangle + E\langle t,y\rangle) -$$
$$-(\tfrac{1}{2}H\langle y,u\rangle + E\langle y,v\rangle)(z,t) + (\tfrac{1}{2}H\langle y,z\rangle + E\langle y,t\rangle)(u,v) -$$
$$-2y(2)(F\langle u,z\rangle + \tfrac{1}{2}H(\langle u,t\rangle + \langle v,z\rangle) + E\langle v,t\rangle) =$$
$$= -\tfrac{1}{2}(-u(1)+v(2))\langle z,y\rangle + y(1)\langle z,u\rangle + \tfrac{1}{2}y(2)\langle z,v\rangle - z(1)\langle y,u\rangle - 2z(2)\langle y,v\rangle -$$
$$-u(2)\langle t,y\rangle - \tfrac{1}{2}y(2)\langle t,u\rangle - t(2)\langle y,u\rangle -$$
$$-y(1)\langle u,z\rangle - \tfrac{1}{2}y(2)\langle u,t\rangle + \tfrac{1}{2}(-z(1)+t(2))\langle u,y\rangle - u(1)\langle z,y\rangle - 2u(2)\langle t,y\rangle +$$
$$+\tfrac{1}{2}y(2)\langle v,z\rangle + z(2)\langle v,y\rangle - v(2)\langle z,y\rangle -$$
$$-\tfrac{1}{2}(-z(1)+t(2))\langle y,u\rangle + z(2)\langle y,v\rangle + \tfrac{1}{2}(-u(1)+v(2))\langle y,z\rangle - u(2)\langle y,t\rangle +$$
$$+2y(1)\langle u,z\rangle + y(2)(\langle u,t\rangle + \langle v,z\rangle) =$$
$$= 2(v(2)\langle y,z\rangle + z(2)\langle v,y\rangle) + 2(u(2)\langle y,t\rangle + t(2)\langle u,y\rangle) + y(2)(\langle u,t\rangle + \langle v,z\rangle) =$$
$$\stackrel{(26)}{=} 3y(2)(\langle u,t\rangle + \langle v,z\rangle).$$

Then it easily follows that $\alpha((u,v),(z,t),E)$, $\alpha((u,v),(z,t),F)$ and $\alpha((u,v),(z,t),H)$ are elements of $\Gamma(\mathcal{M})$.

The case $\alpha((u,v),(z,t),(p,q))$

Finally, we have

$$
\begin{gathered}
E\alpha((u,v),(z,t),(p,q)) = p(E,(u,v),(z,t),(p,q)) = \\
= -\{(z,t)(p,q),E,(u,v)\} - \{(u,v)(p,q),(z,t),E\} + \{E(p,q),(u,v),(z,t)\} = \\
= -\{F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle, E, (u,v)\} - \\
-\{F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle, (z,t), E\} - \{p(2),(u,v),(z,t)\} = \\
= -((F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)E(u,v) - \\
-(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(u,v)E + \\
+2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)\cdot E(u,v)) - \\
-((F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(z,t)E - \\
-(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)E(z,t) + \\
+2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)\cdot(z,t)E) - \\
-(p(2)(u,v)(z,t) - p(2)(z,t)(u,v) + 2p(2)\cdot(u,v)(z,t)) = \\
= (\tfrac{1}{2}H\langle z,p\rangle + \tfrac{1}{2}E(\langle z,q\rangle + \langle t,p\rangle))(u,v) + \\
+(v(1)\langle z,p\rangle - \tfrac{1}{2}u(1)(\langle z,q\rangle + \langle t,p\rangle) - u(2)\langle t,q\rangle)E + \\
+2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)u(2) + \\
(-t(1)\langle u,p\rangle + \tfrac{1}{2}z(1)(\langle u,q\rangle + \langle v,p\rangle) - \tfrac{1}{2}t(2)(\langle u,q\rangle + \langle v,p\rangle) + z(2)\langle v,q\rangle)E + \\
(-\tfrac{1}{2}H\langle u,p\rangle - \tfrac{1}{2}E(\langle u,q\rangle + \langle v,p\rangle))(z,t) - \\
-2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)z(2) - \\
-(\tfrac{1}{2}H\langle p,u\rangle + E\langle p,v\rangle)(z,t) + (\tfrac{1}{2}H\langle p,z\rangle + E\langle p,t\rangle)(u,v) - \\
-2p(2)(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle) = \\
= -\tfrac{1}{2}u(1)\langle z,p\rangle + \tfrac{1}{2}v(2)\langle z,p\rangle - \tfrac{1}{2}u(2)(\langle z,q\rangle + \langle t,q\rangle) + \\
+v(2)\langle z,p\rangle - \tfrac{1}{2}u(2)(\langle z,q\rangle + \langle t,p\rangle) + 2u(1)\langle z,p\rangle + u(2)(\langle z,q\rangle + \langle t,p\rangle) - \\
-t(2)\langle u,p\rangle + \tfrac{1}{2}z(2)(\langle u,q\rangle + \langle v,p\rangle) + \tfrac{1}{2}z(1)\langle u,p\rangle - \tfrac{1}{2}t(2)\langle u,p\rangle + \tfrac{1}{2}z(2)(\langle u,q\rangle + \langle v,p\rangle) - \\
-2z(1)\langle u,p\rangle - z(2)(\langle u,q\rangle + \langle v,p\rangle) + \\
+\tfrac{1}{2}z(1)\langle p,u\rangle - \tfrac{1}{2}t(2)\langle p,u\rangle + z(2)\langle p,v\rangle - \tfrac{1}{2}u(1)\langle p,z\rangle + \tfrac{1}{2}v(2)\langle p,z\rangle - u(2)\langle p,t\rangle + \\
+2p(1)\langle u,z\rangle + p(2)\langle u,t\rangle + p(2)\langle v,z\rangle = \\
= 2(u(1)\langle z,p\rangle + z(1)\langle p,u\rangle + p(1)\langle u,z\rangle) + \\
+t(2)\langle p,u\rangle + p(2)\langle u,t\rangle + u(2)\langle t,p\rangle + \\
+v(2)\langle z,p\rangle + z(2)\langle p,v\rangle + p(2)\langle v,z\rangle \overset{(26)}{=} 0.
\end{gathered}
$$

Also,

$$
\begin{gathered}
F\alpha((u,v),(z,t),(p,q)) = p(F,(u,v),(z,t),(p,q)) = \\
= -\{(z,t)(p,q),F,(u,v)\} - \{(u,v)(p,q),(z,t),F\} + \{F(p,q),(u,v),(z,t)\} = \\
= -\{F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle, F, (u,v)\} - \\
-\{F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle, (z,t), F\} + \{q(1),(u,v),(z,t)\} = \\
= -((F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)F(u,v) - \\
-(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(u,v)F + \\
+2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)\cdot F(u,v)) - \\
-((F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(z,t)F - \\
-(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)F(z,t) + \\
+2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)\cdot (z,t)F) + \\
+q(1)(u,v)(z,t) - q(1)(z,t)(u,v) + 2q(1)\cdot(u,v)(z,t) = \\
= -(\tfrac{1}{2}F(\langle z,q\rangle + \langle t,p\rangle) + \tfrac{1}{2}H\langle t,q\rangle)(u,v) + \\
+(v(1)\langle z,p\rangle - \tfrac{1}{2}u(1)(\langle z,q\rangle + \langle t,p\rangle) + \tfrac{1}{2}v(2)(\langle z,q\rangle + \langle t,p\rangle) - u(2)\langle t,q\rangle)F - \\
-2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)v(1) - \\
-(t(1)\langle u,p\rangle - \tfrac{1}{2}z(1)(\langle u,q\rangle + \langle v,p\rangle) + \tfrac{1}{2}t(2)(\langle u,q\rangle + \langle v,p\rangle) - z(2)\langle v,q\rangle)F + \\
+(\tfrac{1}{2}F(\langle u,q\rangle + \langle v,p\rangle) + \tfrac{1}{2}H\langle v,q\rangle)(z,t) + \\
+2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)t(1) + \\
+(F\langle q,u\rangle + \tfrac{1}{2}H\langle q,v\rangle)(z,t) - (F\langle q,z\rangle + \tfrac{1}{2}H\langle q,t\rangle)(u,v) + \\
+2q(1)(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle) = \\
= -\tfrac{1}{2}v(1)(\langle z,q\rangle + \langle t,p\rangle) + \tfrac{1}{2}u(1)\langle t,q\rangle - \tfrac{1}{2}v(2)\langle t,q\rangle - \\
-\tfrac{1}{2}v(1)(\langle z,q\rangle + \langle t,p\rangle) + u(1)\langle t,q\rangle + \\
+v(1)(\langle z,q\rangle + \langle t,p\rangle) + 2v(2)\langle t,q\rangle) + \\
+\tfrac{1}{2}t(1)(\langle u,q\rangle + \langle v,p\rangle) - z(1)\langle v,q\rangle + \\
+\tfrac{1}{2}t(1)(\langle u,q\rangle + \langle v,p\rangle) - \tfrac{1}{2}z(1)\langle v,q\rangle + \tfrac{1}{2}t(2)\langle v,q\rangle - \\
-t(1)(\langle u,q\rangle + \langle v,p\rangle) - 2t(2)\langle v,q\rangle + \\
+t(1)\langle q,u\rangle - \tfrac{1}{2}z(1)\langle q,v\rangle + \tfrac{1}{2}t(2)\langle q,v\rangle - \\
-v(1)\langle q,z\rangle + \tfrac{1}{2}u(1)\langle q,t\rangle - \tfrac{1}{2}v(2)\langle q,t\rangle + \\
+q(1)(\langle u,t\rangle + \langle v,z\rangle) + 2q(2)\langle v,t\rangle) = \\
= u(1)\langle t,q\rangle + t(1)\langle q,u\rangle + q(1)\langle u,t\rangle + \\
+z(1)\langle q,v\rangle + q(1)\langle v,z\rangle + v(1)\langle z,q\rangle + \\
+2(v(2)\langle t,q\rangle + t(2)\langle q,v\rangle + q(2)\langle v,t\rangle) \overset{(26)}{=} 0.
\end{gathered}
$$

In addition,

$$
\begin{gathered}
H\alpha((u,v),(z,t),(p,q)) = p(H,(u,v),(z,t),(p,q)) = \\
= -\{(z,t)(p,q),H,(u,v)\} - \{(u,v)(p,q),(z,t),H\} + \{H(p,q),(u,v),(z,t)\} = \\
= -\{F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle, H, (u,v)\} - \\
-\{F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle, (z,t), H\} + \{-p(1)+q(2),(u,v),(z,t)\} = \\
= -((F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)H(u,v) - \\
-(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(u,v)H + \\
+2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)\cdot H(u,v)) - \\
-((F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(z,t)H - \\
-(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)H(z,t) + \\
+2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)\cdot(z,t)H) - \\
-(p(1)(u,v)(z,t) - p(1)(z,t)(u,v) + 2p(1)\cdot(u,v)(z,t)) + \\
+q(2)(u,v)(z,t) - q(2)(z,t)(u,v) + 2q(2)\cdot(u,v)(z,t) = \\
= -(-F\langle z,p\rangle + E\langle t,q\rangle)(u,v) + \\
+(v(1)\langle z,p\rangle - \tfrac{1}{2}u(1)(\langle z,q\rangle + \langle t,p\rangle) + \tfrac{1}{2}v(2)(\langle z,q\rangle + \langle t,p\rangle) - u(2)\langle t,q\rangle)H - \\
-2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(-u(1)+v(2)) - \\
-(t(1)\langle u,p\rangle - \tfrac{1}{2}z(1)(\langle u,q\rangle + \langle v,p\rangle) + \tfrac{1}{2}t(2)(\langle u,q\rangle + \langle v,p\rangle) - z(2)\langle v,q\rangle)H + \\
+(-F\langle u,p\rangle + E\langle v,q\rangle)(z,t) - \\
-2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(z(1)-t(2)) - \\
-(F\langle p,u\rangle + \tfrac{1}{2}H\langle p,v\rangle)(z,t) + (F\langle p,z\rangle + \tfrac{1}{2}H\langle p,t\rangle)(u,v) - \\
-2p(1)(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle) + \\
+(\tfrac{1}{2}H\langle q,u\rangle + E\langle q,v\rangle)(z,t) - (\tfrac{1}{2}H\langle q,z\rangle + E\langle q,t\rangle)(u,v) + \\
+2q(2)(F\langle u,z\rangle + \tfrac{1}{2}H\langle u,t\rangle + \tfrac{1}{2}H\langle v,z\rangle + E\langle v,t\rangle) = \\
= v(1)\langle z,p\rangle + u(2)\langle t,q\rangle + \\
+v(1)\langle z,p\rangle - \tfrac{1}{2}u(1)(\langle z,q\rangle + \langle t,p\rangle) - \tfrac{1}{2}v(2)(\langle z,q\rangle + \langle t,p\rangle) + u(2)\langle t,q\rangle - \\
-2v(1)\langle z,p\rangle - u(1)(\langle z,q\rangle + \langle t,p\rangle) - v(2)(\langle z,q\rangle + \langle t,p\rangle) - 2u(2)\langle t,q\rangle - \\
-t(1)\langle u,p\rangle + \tfrac{1}{2}z(1)(\langle u,q\rangle + \langle v,p\rangle) + \tfrac{1}{2}t(2)(\langle u,q\rangle + \langle v,p\rangle) - z(2)\langle v,q\rangle - \\
-t(1)\langle u,p\rangle - z(2)\langle v,q\rangle - \\
2t(1)\langle u,p\rangle + z(1)(\langle u,q\rangle + \langle v,p\rangle) + t(2)(\langle u,q\rangle + \langle v,p\rangle) + 2z(2)\langle v,q\rangle - \\
-t(1)\langle p,u\rangle + \tfrac{1}{2}z(1)\langle p,v\rangle - \tfrac{1}{2}t(2)\langle p,v\rangle + v(1)\langle p,z\rangle - \tfrac{1}{2}u(1)\langle p,t\rangle + \tfrac{1}{2}v(2)\langle p,t\rangle - \\
-p(1)\langle u,t\rangle - p(1)\langle v,z\rangle - 2p(2)\langle v,t\rangle +
\end{gathered}
$$

$$-\tfrac{1}{2}z(1)\langle q,u\rangle + \tfrac{1}{2}t(2)\langle q,u\rangle - z(2)\langle q,v\rangle + \tfrac{1}{2}u(1)\langle q,z\rangle - \tfrac{1}{2}v(2)\langle q,z\rangle + u(2)\langle q,t\rangle +$$
$$-2q(1)\langle u,z\rangle - q(2)\langle u,t\rangle - q(2)\langle v,z\rangle =$$
$$= 2(u(1)\langle q,z\rangle + q(1)\langle z,u\rangle + z(1)\langle u,q\rangle) +$$
$$+2(v(2)\langle p,t\rangle + p(2)\langle t,v\rangle + t(2)\langle v,p\rangle) +$$
$$+u(1)\langle p,t\rangle + p(1)\langle t,u\rangle + t(1)\langle u,p\rangle +$$
$$+z(1)\langle v,p\rangle + v(1)\langle p,z\rangle + p(1)\langle z,v\rangle +$$
$$+z(2)\langle v,q\rangle + v(2)\langle q,z\rangle + q(2)\langle z,v\rangle +$$
$$+u(2)\langle q,t\rangle + q(2)\langle t,u\rangle + t(2)\langle u,q\rangle \overset{(26)}{=} 0.$$

Besides,

$$x(1)\alpha((u,v),(z,t),(p,q)) = p(x(1),(u,v),(z,t),(p,q)) =$$
$$= -\{(z,t)(p,q),x(1),(u,v)\} - \{(u,v)(p,q),(z,t),x(1)\} + \{x(1)(p,q),(u,v),(z,t)\} =$$
$$= -\{F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle, x(1),(u,v)\} -$$
$$-\{F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle,(z,t),x(1)\} +$$
$$+\{F\langle x,p\rangle + \tfrac{1}{2}H\langle x,q\rangle,(u,v),(z,t)\} =$$
$$= -((F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)x(1)(u,v) -$$
$$-(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(u,v)x(1) +$$
$$+2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)\cdot x(1)(u,v)) -$$
$$-((F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(z,t)x(1) -$$
$$-(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)x(1)(z,t) +$$
$$+2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)\cdot(z,t)x(1)) +$$
$$+\{F,(u,v),(z,t)\}\langle x,p\rangle + \tfrac{1}{2}\{H,(u,v),(z,t)\}\langle x,q\rangle =$$
$$\overset{(21)-(24)}{=} (\tfrac{1}{2}x(1)(\langle z,q\rangle + \langle t,p\rangle) + x(2)\langle t,q\rangle)(u,v) +$$
$$+(v(1)\langle z,p\rangle - \tfrac{1}{2}(u(1) - v(2))(\langle z,q\rangle + \langle t,p\rangle) - u(2)\langle t,q\rangle)x(1) -$$
$$-2(F\langle z,p\rangle + \tfrac{1}{2}H(\langle z,q\rangle + \langle t,p\rangle) + E\langle t,q\rangle)(F\langle x,u\rangle + \tfrac{1}{2}H\langle x,v\rangle) -$$
$$-(t(1)\langle u,p\rangle - \tfrac{1}{2}(z(1) - t(2))(\langle u,q\rangle + \langle v,p\rangle) - z(2)\langle v,q\rangle)x(1) -$$
$$-(\tfrac{1}{2}x(1)(\langle u,q\rangle + \langle v,p\rangle) + x(2)\langle v,q\rangle)(z,t) -$$
$$-2(F\langle u,p\rangle + \tfrac{1}{2}H(\langle u,q\rangle + \langle v,p\rangle) + E\langle v,q\rangle)(F\langle z,x\rangle + \tfrac{1}{2}H\langle t,x\rangle) =$$
$$= \tfrac{1}{2}F\langle x,u\rangle(\langle z,q\rangle + \langle t,p\rangle) + \tfrac{1}{4}H\langle x,v\rangle(\langle z,q\rangle + \langle t,p\rangle) + (\tfrac{1}{2}H\langle x,u\rangle + E\langle x,v\rangle)\langle t,q\rangle +$$
$$+F\langle v,x\rangle\langle z,p\rangle - \tfrac{1}{2}(F\langle u,x\rangle - \tfrac{1}{2}H\langle v,x\rangle)(\langle z,q\rangle + \langle t,p\rangle) - \tfrac{1}{2}H\langle u,x\rangle\langle t,q\rangle +$$
$$+F\langle z,p\rangle\langle x,v\rangle - F\langle x,u\rangle(\langle z,q\rangle + \langle t,p\rangle) - H\langle t,q\rangle\langle x,u\rangle - E\langle t,q\rangle\langle x,v\rangle -$$

$$-F\langle t,x\rangle\langle u,p\rangle+\tfrac{1}{2}(F\langle z,x\rangle-\tfrac{1}{2}H\langle t,x\rangle)(\langle u,q\rangle+\langle v,p\rangle)+\tfrac{1}{2}H\langle z,x\rangle\langle v,q\rangle-$$
$$-\tfrac{1}{2}(F\langle x,z\rangle+\tfrac{1}{2}H\langle x,t\rangle)(\langle u,q\rangle+\langle v,p\rangle)-\tfrac{1}{2}H\langle x,z\rangle\langle v,q\rangle-E\langle x,t\rangle\langle v,q\rangle+$$
$$+F\langle u,p\rangle\langle t,x\rangle-F(\langle u,q\rangle+\langle v,p\rangle)\langle z,x\rangle-H\langle v,q\rangle\langle z,x\rangle-E\langle v,q\rangle\langle t,x\rangle=0$$

and

$$y(2)\alpha((u,v),(z,t),(p,q))=p(y(2),(u,v),(z,t),(p,q))=$$
$$=-\{(z,t)(p,q),y(2),(u,v)\}-\{(u,v)(p,q),(z,t),y(2)\}+\{y(2)(p,q),(u,v),(z,t)\}=$$
$$=-\{F\langle z,p\rangle+\tfrac{1}{2}H(\langle z,q\rangle+\langle t,p\rangle)+E\langle t,q\rangle,y(2),(u,v)\}-$$
$$-\{F\langle u,p\rangle+\tfrac{1}{2}H(\langle u,q\rangle+\langle v,p\rangle)+E\langle v,q\rangle,(z,t),y(2)\}+$$
$$+\{\tfrac{1}{2}H\langle y,p\rangle+E\langle y,q\rangle,(u,v),(z,t)\}=$$
$$=-((F\langle z,p\rangle+\tfrac{1}{2}H(\langle z,q\rangle+\langle t,p\rangle)+E\langle t,q\rangle)y(2)(u,v)-$$
$$-(F\langle z,p\rangle+\tfrac{1}{2}H(\langle z,q\rangle+\langle t,p\rangle)+E\langle t,q\rangle)(u,v)y(2)+$$
$$+2(F\langle z,p\rangle+\tfrac{1}{2}H(\langle z,q\rangle+\langle t,p\rangle)+E\langle t,q\rangle)\cdot y(2)(u,v))-$$
$$-((F\langle u,p\rangle+\tfrac{1}{2}H(\langle u,q\rangle+\langle v,p\rangle)+E\langle v,q\rangle)(z,t)y(2)-$$
$$-(F\langle u,p\rangle+\tfrac{1}{2}H(\langle u,q\rangle+\langle v,p\rangle)+E\langle v,q\rangle)y(2)(z,t)+$$
$$+2(F\langle u,p\rangle+\tfrac{1}{2}H(\langle u,q\rangle+\langle v,p\rangle)+E\langle v,q\rangle)\cdot(z,t)y(2))+$$
$$+\tfrac{1}{2}\{H,(u,v),(z,t)\}\langle y,p\rangle+\{E,(u,v),(z,t)\}\langle y,q\rangle=$$
$$\overset{(21)-(24)}{=}-(y(1)\langle z,p\rangle+\tfrac{1}{2}y(2)(\langle z,q\rangle+\langle t,p\rangle))(u,v)+$$
$$+(v(1)\langle z,p\rangle-\tfrac{1}{2}(u(1)-v(2))(\langle z,q\rangle+\langle t,p\rangle)-u(2)\langle t,q\rangle)y(2)-$$
$$-2(F\langle z,p\rangle+\tfrac{1}{2}H(\langle z,q\rangle+\langle t,p\rangle)+E\langle t,q\rangle)(\tfrac{1}{2}H\langle y,u\rangle+E\langle y,v\rangle)-$$
$$-(t(1)\langle u,p\rangle-\tfrac{1}{2}(z(1)-t(2))(\langle u,q\rangle+\langle v,p\rangle)-z(2)\langle v,q\rangle)y(2)+$$
$$+(y(1)\langle u,p\rangle+\tfrac{1}{2}y(2)(\langle u,q\rangle+\langle v,p\rangle))(z,t)-$$
$$-2(F\langle u,p\rangle+\tfrac{1}{2}H(\langle u,q\rangle+\langle v,p\rangle)+E\langle v,q\rangle)(\tfrac{1}{2}H\langle z,y\rangle+E\langle t,y\rangle)=$$
$$=-(F\langle y,u\rangle+\tfrac{1}{2}H\langle y,v\rangle)\langle z,p\rangle-\tfrac{1}{2}(\tfrac{1}{2}H\langle y,u\rangle+E\langle y,v\rangle)(\langle z,q\rangle+\langle t,p\rangle))+$$
$$+\tfrac{1}{2}H\langle v,y\rangle\langle z,p\rangle-\tfrac{1}{2}(\tfrac{1}{2}H\langle u,y\rangle-E\langle v,y\rangle)(\langle z,q\rangle+\langle t,p\rangle)-E\langle u,y\rangle\langle t,q\rangle+$$
$$+F\langle z,p\rangle\langle y,u\rangle+H\langle z,p\rangle\langle y,v\rangle+E(\langle z,q\rangle+\langle t,p\rangle)\langle y,v\rangle-E\langle t,q\rangle\langle y,u\rangle-$$
$$-\tfrac{1}{2}H\langle t,y\rangle\langle u,p\rangle+\tfrac{1}{2}(\tfrac{1}{2}H\langle z,y\rangle-E\langle t,y\rangle)(\langle u,q\rangle+\langle v,p\rangle)+E\langle z,y\rangle\langle v,q\rangle+$$
$$+(F\langle y,z\rangle+\tfrac{1}{2}H\langle y,t\rangle)\langle u,p\rangle+\tfrac{1}{2}(\tfrac{1}{2}H\langle y,z\rangle+E\langle y,t\rangle)(\langle u,q\rangle+\langle v,p\rangle)+$$
$$+F\langle u,p\rangle\langle z,y\rangle+H\langle u,p\rangle\langle t,y\rangle+E(\langle u,q\rangle+\langle v,p\rangle)\langle t,y\rangle-E\langle v,q\rangle\langle z,y\rangle=0.$$

Thus $\alpha((u,v),(z,t),(p,q))|_{\mathcal{M}}=0$; so $\alpha((u,v),(z,t),(p,q))$ belongs to $\Gamma(\mathcal{M})$.

Consequently, from the above cases we get that $\mathcal{M}$ belongs to the variety $\mathcal{H}$. Therefore, the theorem is proved. $\square$

We know that the commutator in any alternative algebra satisfies the identities (1), so every Lie algebra is a Malcev algebra. The speciality problem for Malcev algebras asks if any Malcev algebra is isomorphic to a subalgebra of the commutator algebra of some alternative algebra. In this case, we have the following.

Corollary 3.15. *Every Malcev algebra $\mathcal{M}$ in $\mathcal{H}$ containing $L \cong \mathfrak{sl}_2$ with $mL \neq 0$ for any $0 \neq m \in \mathcal{M}$ is special.*

3.2. Examples

3.2.1. Exceptional Malcev algebras

Let $\mathcal{B}$ be a unital associative commutative algebra over a field of characteristic $\neq 2, 3$ and consider the full 2×2 matrix algebra matrices $M_2(\mathcal{B})$ over $\mathcal{B}$ with the following basis:

$$I = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Let $\mathcal{M} = \mathcal{M}_7(\mathcal{B})$ be the exceptional Malcev algebra over $\mathcal{B}$. In this case,

$$\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus vM_2(\mathcal{B})$$

where $N_{\mathcal{M}} = \mathfrak{sl}_2(\mathcal{B})$, $J_{\mathcal{M}} = vM_2(\mathcal{B})$, with the product defined by

$$X \cdot Y = XY - YX, \quad X \cdot vA = v\left(X^* A - XA\right), \quad vA \cdot vB = BA^* - AB^*, \tag{33}$$

where $X, Y \in \mathfrak{sl}_2(\mathcal{B})$, $A, B \in M_2(\mathcal{B})$ and $A \mapsto A^*$ is the symplectic involution:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Take $V = \mathcal{B}^2 = \{(a, b) \mid a, b \in \mathcal{B}\}$, $(a, b)(1) = v\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$, $(a, b)(2) = v\begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix}$. Then using (33) it is easy to see that $\{(a, b)(1), (a, b)(2)\}$ satisfies the relations (25). Therefore, we have $\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V(1) \oplus V(2)$, with $\langle (a, b), (c, d) \rangle = -4\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In fact, by (28)

$$\begin{aligned}F\langle(a,b),(c,d)\rangle &= (a,b)(1)\cdot(c,d)(1)\\ &= v\begin{pmatrix}0 & 0\\ a & b\end{pmatrix}\cdot v\begin{pmatrix}0 & 0\\ c & d\end{pmatrix}\\ &\overset{(33)}{=}\begin{pmatrix}0 & 0\\ c & d\end{pmatrix}\begin{pmatrix}b & 0\\ -a & 0\end{pmatrix}-\begin{pmatrix}0 & 0\\ a & b\end{pmatrix}\begin{pmatrix}d & 0\\ -c & 0\end{pmatrix}=-4F\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\end{aligned}$$

So $F(\langle(a,b),(c,d)\rangle+4\det\begin{pmatrix}a & b\\ c & d\end{pmatrix})=0$ implies $L(\langle(a,b),(c,d)\rangle+4\det\begin{pmatrix}a & b\\ c & d\end{pmatrix})=0$. Thus,

$$\langle(a,b),(c,d)\rangle=-4\mathrm{det}\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Now for any $u=(a,b), v=(c,d), w=(e,f)\in V$ we have

$$\begin{aligned}&w\langle u,v\rangle+u\langle v,w\rangle+v\langle w,u\rangle\\ &\quad=-4(e,f)\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}-4(a,b)\det\begin{pmatrix}c & d\\ e & f\end{pmatrix}-4(c,d)\det\begin{pmatrix}e & f\\ a & b\end{pmatrix}\\ &\quad=-4(e,f)(ad-bc)-4(a,b)(cf-de)-4(c,d)(eb-fa)=(0,0);\end{aligned}$$

hence, $\mathcal{M}_7(\mathcal{B})$ satisfies (26).

The following proposition gives a specific certain condition to obtain a coordinatization theorem for Malcev algebras containing $L=\mathfrak{sl}_2(\mathbb{F})$.

Proposition 3.16. *The algebra $\mathcal{M}=\mathfrak{sl}_2(\mathcal{B})\oplus V^2$ from Theorem 3.14 is isomorphic to the exceptional Malcev algebra $\mathcal{M}_7(\mathcal{B})$ if and only if there exist $u,v\in V$ such that $\langle u,v\rangle=1$.*

Proof. We have already checked that the algebra $\mathcal{M}_7(\mathcal{B})$ has the form $\mathfrak{sl}_2(\mathcal{B})\oplus V^2$. It is observed that $\langle u,v\rangle=1$ for $u=(\frac{1}{2},0)$, $v=(0,-\frac{1}{2})\in V$.

Let now $\mathcal{A}=\mathfrak{sl}_2(\mathbb{F})+V_2(u)+V_2(v)$ be such that there exist $u,v\in V$ with $\langle u,v\rangle=1$. It follows from Proposition 3.9 and its proof that $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ isomorphic to the 7-dimensional exceptional Malcev algebra $\mathcal{M}_7(\mathbb{F})$ with $m\mathcal{A}\neq 0$ for any $0\neq m\in\mathcal{M}$ because $mL\neq 0$ for any $0\neq m\in\mathcal{M}$. Therefore, by [22] (Theorem 3.1), $\mathcal{M}\cong\mathcal{M}_7(\mathcal{U})$ for a certain associative commutative algebra $\mathcal{U}$. It follows from (26) that $V=\mathcal{B}\cdot V_2(u)+\mathcal{B}\cdot V_2(v)$ and $\mathcal{U}=\mathcal{B}$. Thus, the proposition is proved. □

3.2.2. Algebras obtained by (commutative) Cayley–Dickson process

Note that if the mapping $\langle\cdot,\cdot\rangle : V^2\longrightarrow\mathcal{B}$ is trivial, then the algebra $\mathcal{M}$ is just a split null extension of the algebra $\mathfrak{sl}_2(\mathcal{B})$ by a bimodule V^2. In this case, V may be an arbitrary commutative $\mathcal{B}$-bimodule. For instance, when $\mathcal{B} = \mathbb{F}$ and $V = \mathbb{F}$ we get in this way the algebra $\mathcal{M}_5(\mathbb{F}) = \mathfrak{sl}_2(\mathbb{F}) \oplus V_2(u)$ which is a non-Lie Malcev algebra of dimension five.

Example 3.17. The 5-dimensional non-Lie Malcev algebra $\mathcal{M}_5(\mathbb{F}) = \mathfrak{sl}_2(\mathbb{F}) \oplus V_2(u)$ that contains $\mathfrak{sl}_2(\mathbb{F})$ has the following multiplication table

$\cdot$	E	F	H	$u(1)$	$u(2)$
E	0	$H/2$	E	$-u(2)$	0
F	$-H/2$	0	$-F$	0	$u(1)$
H	$-E$	F	0	$-u(1)$	$u(2)$
$u(1)$	$u(2)$	0	$u(1)$	0	0
$u(2)$	0	$-u(1)$	$-u(2)$	0	0

where $\{u(1), u(2)\}$ is the basis of $V_2(u)$.

If the mapping $\langle\cdot,\cdot\rangle : V^2 \to \mathcal{B}$ is not trivial, then by (26) the rank of V as a $\mathcal{B}$-bimodule is less than 3. Observe that the left side of (26) is $\mathcal{B}$-multilinear and skew-symmetric on u, v, w. Therefore, it holds when $\Lambda^3 (V_{\mathcal{B}}) = 0$. In particular, it holds if the rank of V is less than or equal to 2. If $V \subseteq \mathcal{B} \cdot x$ then the mapping $\langle\cdot,\cdot\rangle$ is trivial by skew-symmetry. Let us now consider the case when V is a 2-generated $\mathcal{B}$-module.

Let $\mathcal{U}$ be a unital associative commutative algebra over a field of characteristic $\neq 2, 3$ and $\alpha \in \mathcal{U}$. Denote by $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{U}), \alpha)$ the algebra $\mathfrak{sl}_2(\mathcal{U}) \oplus vM_2(\mathcal{U})$ with a product defined by the following analog of (33):

$$X \cdot Y = XY - YX, \quad X \cdot vA = v\,(X^*A - XA)\,, \quad vA \cdot vB = \alpha(BA^* - AB^*), \tag{34}$$

where $X, Y \in \mathfrak{sl}_2(\mathcal{B})$, $A, B \in M_2(\mathcal{B})$ and $A \mapsto A^*$ is the symplectic involution.

The algebra $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{U}), \alpha)$ is a Malcev algebra that contains $\mathfrak{sl}_2(\mathcal{U})$. Here

$$\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{U})), \alpha) = \mathrm{CD}\,(M_2(\mathcal{B}), \alpha)^{(-)}\,/\mathbb{F} \cdot 2I$$

where $2I$ is a matriz identity of $M_2(\mathcal{U})$. The algebra $\mathrm{CD}\,(M_2(\mathcal{U}), \alpha) = M_2(\mathcal{U}) \oplus vM_2(\mathcal{U})$ is an alternative algebra containing $M_2(\mathcal{U})$ with the same identity element, which by [24] we will call it the *algebra obtained from $M_2(\mathcal{U})$ by the Cayley–Dickson process* with parameter α. The algebra $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{U}), \alpha)$ is an exceptional Malcev algebra if and only if the parameter α is invertible in $\mathcal{U}$.

Theorem 3.18. *Let $\mathcal{B}$ be a unital associative commutative algebra, let $V = \mathcal{B}^2$, and let $\langle\cdot,\cdot\rangle : V^2 \to \mathcal{B}$ be a skew-symmetric $\mathcal{B}$-bilinear mapping. Then the algebra $\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V^2$ is isomorphic to the algebra $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{B}), \alpha)$, where $\alpha = -\langle(\frac{1}{2}, 0), (0, \frac{1}{2})\rangle$. Conversely, every algebra $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(\mathcal{U}), \alpha)$ has this form.*

Proof. Let $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(\mathfrak{sl}_2(\mathcal{U}), \alpha)$. Take $V = \mathcal{U}^2 = \{(a,b) \mid a,b \in \mathcal{U}\}, (a,b)(1) = v\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$, and $(a,b)(2) = v\begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix} \in vM_2(\mathcal{U})$. Then we have, as before, $\widetilde{\mathcal{M}} = \mathfrak{sl}_2(\mathcal{U}) \oplus V(1) \oplus V(2)$, with $\langle (a,b),(c,d)\rangle = -4\alpha \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular, $\langle (\frac{1}{2}, 0), (0, \frac{1}{2})\rangle = -\alpha$.

Conversely, let $\mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V^2$, where $V \cong \mathcal{B}^2$ and $\langle (\frac{1}{2}, 0), (0, \frac{1}{2})\rangle = -\alpha$. Define the mapping $\varphi : V^2 = V(1) \oplus V(2) \to vM_2(\mathcal{B}) \subset \widetilde{\mathcal{M}}(\mathfrak{sl}_2(\mathcal{B}), \alpha)$ by sending, for any $a, b \in \mathcal{B}$.

$$(a,b)(1) \mapsto v\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad (a,b)(2) \mapsto v\begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix}$$

It is easy to see that φ is an isomorphism of Malcev $\mathfrak{sl}_2(\mathcal{B})$-modules. Moreover, let $x = (a,b)$, $y = (c,d) \in V = \mathcal{B}^2$, then we have

$$\begin{aligned}\langle x, y\rangle = \langle (a,b),(c,d)\rangle &= \langle a(1,0) + b(0,1), c(1,0) + d(0,1)\rangle \\ &= (ad - bc)\langle (1,0),(0,1)\rangle = -4\alpha(ad - bc).\end{aligned}$$

Let $z = (e,f)$, $t = (g,h) \in V$; then we have by (32),

$$\begin{aligned}(x,y)(z,t) &= \frac{1}{2}\begin{pmatrix} -\frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) & -\langle y,t\rangle \\ \langle x,z\rangle & \frac{1}{2}(\langle x,t\rangle + \langle y,z\rangle) \end{pmatrix} \\ &= \begin{pmatrix} \alpha(ah - bg + cf - de) & 2\alpha(ch - dg) \\ -2\alpha(af - be) & -\alpha(ah - bg + cf - de) \end{pmatrix}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(x,y) \cdot \varphi(z,t) &= v\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &\overset{(34)}{=} \alpha[\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}] \\ &= \alpha\begin{pmatrix} -bg + ah + cf - de & 2(ch - dg) \\ 2(-af + be) & de - cf - ah + bg \end{pmatrix}.\end{aligned}$$

Therefore, the mapping

$$\mathrm{id} + \varphi : \mathcal{M} = \mathfrak{sl}_2(\mathcal{B}) \oplus V^2 \longrightarrow \widetilde{\mathcal{M}}(\mathfrak{sl}_2(\mathcal{U}), \alpha) = \mathfrak{sl}_2(\mathcal{B}) \oplus vM_2(\mathcal{B})$$

is an isomorphism.

The theorem is proved. $\square$

Example 3.19. We have the 7-dimensional exceptional Malcev algebra $\mathcal{M}_7(\mathbb{F}) = \mathfrak{sl}_2(\mathbb{F}) \oplus V_2(u) \oplus V_2(v)$ that contains $\mathfrak{sl}_2(\mathbb{F})$. The multiplication table of $\mathcal{M}_7(\mathbb{F})$ is

$\cdot$	E	F	H	$u(1)$	$u(2)$	$v(1)$	$v(2)$
E	0	$H/2$	E	$-u(2)$	0	$-v(2)$	0
F	$-H/2$	0	$-F$	0	$u(1)$	0	$v(1)$
H	$-E$	F	0	$-u(1)$	$u(2)$	$-v(1)$	$v(2)$
$u(1)$	$u(2)$	0	$u(1)$	0	0	$F\langle u,v\rangle$	$H\langle u,v\rangle/2$
$u(2)$	0	$-u(1)$	$-u(2)$	0	0	$H\langle u,v\rangle/2$	$E\langle u,v\rangle$
$v(1)$	$v(2)$	0	$v(1)$	$-F\langle u,v\rangle$	$-H\langle u,v\rangle/2$	0	0
$v(2)$	0	$-v(1)$	$-v(2)$	$-H\langle u,v\rangle/2$	$-E\langle u,v\rangle$	0	0

for $\langle u, v\rangle \in \mathbb{F}$, where $\{u(1), u(2)\}$ and $\{v(1), v(2)\}$ are basis of $V_2(u)$ and $V_2(v)$, respectively.

3.3. The general case

We can easily drop the assumption that $mL \neq 0$ for any $0 \neq m$ from $\mathcal{M}$.

Let $\mathcal{M}$ be a Malcev algebra in $\mathcal{H}$ containing $L = \mathfrak{sl}_2(\mathbb{F})$. Then, we can consider $\mathcal{M}$ as a Malcev $\mathcal{H}$-module over L and by Corollary 3.2 (i), we have

$$\mathcal{M} = \mathrm{Ann}_{\mathcal{M}} L \oplus \widehat{N}_{\mathcal{M}} \oplus J_{\mathcal{M}},$$

where $\mathrm{Ann}_{\mathcal{M}} L = \{m \in \mathcal{M} : mL = 0\}$,

$$\widehat{N}_{\mathcal{M}} = \sum_{0 \neq \alpha \in \alpha(\mathcal{M},L,L)} \oplus L\alpha = \sum_i \oplus \overline{\mathfrak{sl}_2(\mathbb{F})}_i,$$

and $J_{\mathcal{M}} = \sum_i \oplus V_{2i}$, where V_{2i} is a 2-dimensional non-Lie Malcev module for L.

Lemma 3.20. $\mathrm{Ann}_{\mathcal{M}} L$ *is a subalgebra of* $\mathcal{M}$, $(\mathrm{Ann}_{\mathcal{M}} L)\widehat{N}_{\mathcal{M}} = 0$, *and* $(\mathrm{Ann}_{\mathcal{M}} L)J_{\mathcal{M}} \subseteq J_{\mathcal{M}}$.

Proof. Let $m, m' \in \mathrm{Ann}_{\mathcal{M}} L$, then

$$(mm')L = (mm')L^2 = (mm')(LL) \overset{(2)}{=} 0.$$

Thus $(\mathrm{Ann}_{\mathcal{M}} L)^2 \subseteq \mathrm{Ann}_{\mathcal{M}} L$ and $\mathrm{Ann}_{\mathcal{M}} L$ is a subalgebra of $\mathcal{M}$. Furthermore, since $\widehat{N}_{\mathcal{M}} = \sum_i \oplus \overline{\mathfrak{sl}_2(\mathbb{F})}_i$, then

$$(\mathrm{Ann}_{\mathcal{M}} L)\widehat{N}_{\mathcal{M}} = (\mathrm{Ann}_{\mathcal{M}} L) \sum_i \oplus \overline{\mathfrak{sl}_2(\mathbb{F})}_i = 0$$

and $(\mathrm{Ann}_{\mathcal{M}} L)\widehat{N}_{\mathcal{M}} = 0$.

Now as $J_{\mathcal{M}} = V(1) \oplus V(2)$, we will prove that $(\mathrm{Ann}_{\mathcal{M}} L)V_2(u) \subseteq J_{\mathcal{M}}$ for any $u \in V = V(1)$. Consider $m \in \mathrm{Ann}_{\mathcal{M}} L$, then

$$\begin{aligned} -u(1)mF = u(1)m \cdot FH &\overset{(2)}{=} u(1)FmH + FmHu(1) + mHu(1)F + Hu(1)Fm \\ &\overset{(25)}{=} -u(1)Fm = 0, \end{aligned}$$

we get

$$u(1)mF = 0. \tag{35}$$

Also,

$$\begin{aligned} -u(2)mF = u(2)m \cdot FH &\overset{(2)}{=} u(2)FmH + FmHu(2) + mHu(2)F + Hu(2)Fm \\ &= -u(1)mH + u(2)Fm = -u(1)mH - u(1)m \end{aligned}$$

and

$$u(2)mF = u(1)mH + u(1)m. \tag{36}$$

Similarly,

$$\begin{aligned} \frac{1}{2}u(1)mH = u(1)m \cdot EF &\overset{(2)}{=} u(1)EmF + EmFu(1) + mFu(1)E + Fu(1)Em = u(2)mF \\ &\overset{(36)}{=} u(1)mH + u(1)m, \end{aligned}$$

then

$$u(1)mH = -2u(1)m \tag{37}$$

and replacing (37) in (36), we obtain

$$u(2)mF = -u(1)m. \tag{38}$$

Further,

$$u(2)mE = u(2)m \cdot EH \overset{(2)}{=} u(2)EmH + EmHu(2) + mHu(2)E + Hu(2)Em = u(2)Em = 0;$$

thus

$$u(2)mE = 0. \tag{39}$$

In addition,

$$u(1)mE = u(1)m \cdot EH \overset{(2)}{=} u(1)EmH + EmHu(1) + mHu(1)E + Hu(1)Em$$
$$= u(2)mH - u(1)Em = u(2)mH - u(2)m,$$

so

$$u(1)mE = u(2)mH - u(2)m. \tag{40}$$

Finally,

$$\frac{1}{2}u(2)mH = u(2)m \cdot EF \overset{(2)}{=} u(2)EmF + EmFu(2) + mFu(2)E + Fu(2)Em$$
$$= u(1)Em = u(2)m,$$

then

$$u(2)mH = 2u(2)m \tag{41}$$

and replacing (41) in (40), we obtain

$$u(1)mE = u(2)m. \tag{42}$$

The equations (35), (37), (38), (39), (41) and (42) give

$$u(1)mH = -2u(1)m, \;\; u(1)mF = 0, \;\; u(1)mE = u(2)m$$
$$u(2)mH = 2u(2)m, \;\; u(2)mF = -u(1)m, \;\; u(2)mE = 0.$$

Denote $u'(1) = u(1)m$, $u'(2) = u(2)m$, then

$$u'(1)H = -2u'(1), \;\; u'(1)F = 0, \;\; u'(1)E = u'(2),$$
$$u'(2)H = 2u'(2), \;\; u'(2)F = -u'(1), \;\; u'(2)E = 0.$$

Thus, the set $\{u'(1), u'(2)\}$ form a basis of an L-module of type V_2; so $(\mathrm{Ann}_{\mathcal{M}} L)V_2(u) \subseteq J_{\mathcal{M}}$, then $(\mathrm{Ann}_{\mathcal{M}} L)J_{\mathcal{M}} \subseteq J_{\mathcal{M}}$.

The lemma is proved. □

Corollary 3.21. *The decomposition* $\mathcal{M} = (\mathrm{Ann}_{\mathcal{M}} L \oplus \widehat{N}_{\mathcal{M}}) \oplus J_{\mathcal{M}}$ *is a* $\mathbf{Z}_2$*-graded algebra with* $\mathcal{M}_0 = \mathrm{Ann}_{\mathcal{M}} L \oplus \widehat{N}_{\mathcal{M}}$, $\mathcal{M}_1 = J_{\mathcal{M}}$.

Proof. It is clear that $\mathrm{Ann}_{\mathcal{M}} L$ is a Lie L-module. Moreover, from Lemma 3.7 we know that $J_{\mathcal{M}}^2$ is a Lie L-module, then $J_{\mathcal{M}}^2 \subseteq \mathrm{Ann}_{\mathcal{M}} L \oplus \widehat{N}_{\mathcal{M}}$. Thus, the result is a consequence of Lemmas 3.3, 3.20 and Corollary 3.5.

The corollary is proved. □

Furthermore, as already mentioned, from Lemma 3.7 we have $J^2_{\mathcal{M}} \subseteq \mathrm{Ann}_{\mathcal{M}}L \oplus \widehat{N}_{\mathcal{M}}$. A similar process to the proof of Proposition 3.9, leads us to

$$u(1)v(1) = m + F\langle u, v\rangle,$$

$$u(1)v(2) = u(2)v(1) = m + \frac{1}{2}H\langle u, v\rangle,$$

$$u(2)v(2) = m + E\langle u, v\rangle,$$

for some $m \in \mathrm{Ann}_{\mathcal{M}}L$ and $\langle u, v\rangle \in U$, where U is a unital commutative associative algebra. But from (7), $u(1)v(2)F = \frac{1}{2}u(1)v(1)$, then

$$(m + \frac{1}{2}H\langle u, v\rangle)F = \frac{1}{2}(m + F\langle u, v\rangle),$$

implies $m = 0$. Thus, $J^2_{\mathcal{M}} \subseteq \widehat{N}_{\mathcal{M}}$; so by Lemma 3.3 and Corollary 3.5, $\widehat{N}_{\mathcal{M}} \oplus J_{\mathcal{M}} = \mathfrak{sl}_2(U) \oplus V(1) \oplus V(2)$ is a subalgebra of $\mathcal{M}$.

As $\widehat{N}_{\mathcal{M}} \oplus J_{\mathcal{M}}$ is a subalgebra, we can apply to $\widehat{N}_{\mathcal{M}} \oplus J_{\mathcal{M}}$ the results obtained in subsection 3.1.3. Thus, we can write

$$\mathcal{M} = \mathrm{Ann}_{\mathcal{M}}L \oplus (\mathfrak{sl}_2(U) \oplus V^2),$$

where $\widehat{N}_{\mathcal{M}} \cong \mathfrak{sl}_2(U)$, $J_{\mathcal{M}} = V^2$ and U satisfy

$$U \subseteq \Gamma(\mathfrak{sl}_2(U) \oplus V^2)$$

and V is a commutative U-bimodule. In addition, there exists a U-bilinear skew-symmetric mapping $\langle\cdot,\cdot\rangle : V \times V \to U$ such that $\langle V, V\rangle \subseteq U$ and formula (26) holds for any $u, v, w \in V$. Moreover, in $\mathfrak{sl}_2(U) \oplus V^2$ is defined the product (32).

Therefore, we have our main results of this section.

Proposition 3.22. *The Malcev algebra* $\mathcal{M} = \mathrm{Ann}_{\mathcal{M}}L \oplus (\mathfrak{sl}_2(U) \oplus V^2)$ *in* $\mathcal{H}$ *containing* $\mathfrak{sl}_2(\mathbb{F})$ *is isomorphic to* $\mathrm{Ann}_{\mathcal{M}}L \oplus \mathcal{M}_7(U)$ *if and only if there exist* $\langle u, v\rangle = 1$ *for* $u = (\frac{1}{2}, 0), v = (0, -\frac{1}{2}) \in V$.

Proof. From Proposition 3.16, the subalgebra $\mathfrak{sl}_2(U) \oplus V^2$ is isomorphic to the exceptional Malcev algebra $\mathcal{M}_7(U)$ if and only if there exist $\langle u, v\rangle = 1$ for $u = (\frac{1}{2}, 0), v = (0, -\frac{1}{2}) \in V$.

The proposition is proved. □

Finally, from the Cayley–Dickson process we get the following result.

Theorem 3.23. *The Malcev algebra* $\mathcal{M} = \mathrm{Ann}_{\mathcal{M}}L \oplus (\mathfrak{sl}_2(U) \oplus V^2)$ *in* $\mathcal{H}$ *containing* $\mathfrak{sl}_2(\mathbb{F})$ *is isomorphic to* $\mathrm{Ann}_{\mathcal{M}}L \oplus \widetilde{\mathcal{M}}(\mathfrak{sl}_2(U), \alpha)$, *where* $\alpha = -\langle(\frac{1}{2}, 0), (0, \frac{1}{2})\rangle$.

Proof. By Theorem 3.18, the subalgebra $\mathfrak{sl}_2(U) \oplus V^2$ is isomorphic to the Malcev algebra $\widetilde{\mathcal{M}}\,(\mathfrak{sl}_2(U), \alpha)$, where $\alpha = -\langle(\frac{1}{2}, 0), (0, \frac{1}{2})\rangle$.

The theorem is proved. $\square$

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Declaration of competing interest

The author states that there is no conflict of interest.

Data availability

Data will be made available on request.

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